

# RELATIONAL QUADRILATERALLAND. I CONFIGURATION SPACE COORDINATES

Edward Anderson<sup>1</sup>

<sup>1</sup> *Astroparticule et Cosmologie, Université Paris 7 Diderot*

## Abstract

Relational particle models (RPM's) are toy models of many aspects of GR in geometrodynamical form. They are suitable as toy models for studying 1) strategies for the problem of time in quantum gravity, in particular timeless, semiclassical, histories and observables approaches and combinations of these. 2) Various other quantum-cosmological issues: structure formation/inhomogeneity, the significance of uniform states... They are relational in that only relative ratios of separations, relative angles and relative times are significant; more widely, this is a 'Leibniz–Mach–Barbour' brand of relationalism. My recent review of RPM's mostly used the relational triangle as an example; this series of papers extends that to the relational quadrilateral. The relational quadrilateral is more useful in these respects than previously investigated simpler RPM's via simultaneously possessing linear constraints and nontrivial subsystems; also its configuration space is now a nontrivial complex-projective space. This paper studies quadrilateralland's configuration space. In particular, what the relational quadrilateral counterparts of triangleand's A) Dragt-type coordinates (ellipticity, anisoscelesness, and triangle area, which is also a democracy invariant), B) subsystem-split parabolic coordinates and C) the most blockwise-simple coordinates (spherical polars). These were key to unlocking the dynamics, QM and problem of time calculations for the triangle, and their counterparts turn out to be likewise for the quadrilateral in Papers II, III and IV respectively. I show these are now A) a hexuplet of shape coordinates (which now exclude the democracy invariant square root of the sum of squares of areas), B) a linear combination of these termed Kuiper coordinates, and C) the Gibbons–Pope-type coordinates. Each of these is given a lucid new interpretation in terms of quadrilaterals. I furthermore investigate qualitatively-significant regions of the configuration space of quadrilaterals, in anticipation of timeless and Halliwell-type combined Problem of Time strategies and of uniformity in Quantum Cosmology.

PACS: 04.60Kz.

Keywords: Quantum Cosmology, Gravity.

<sup>1</sup> edward.anderson@apc.univ-paris7.fr (Work started at DAMTP Cambridge and continued at UAM Madrid)

# 1 Introduction

## 1.1 RPM's, relationalism and GR

Relational particle mechanics (RPM) are mechanics in which only relative times, relative angles and (ratios of) relative separations are physically meaningful. Scaled RPM was first set up in [1] by Barbour and Bertotti and pure-shape (i.e. scalefree, and so involving just ratios) RPM was first set up in [2] by Barbour. These theories are relational in Leibniz–Mach–Barbour's sense as opposed to Rovelli's (see [3, 4, 5, 6] and especially [7, 8]), via obeying the following postulates.

1) RPM's are *temporally relational* [1, 11, 12]. I.e. they do not possess any meaningful primary notion of time for the whole (model) universe. This is implemented by using actions that are manifestly parametrization-irrelevant while also being free of extraneous time-related variables [such as Newtonian time or the lapse in General Relativity (GR)]. For RPM's, such actions are of the Jacobi type [9] form,<sup>1</sup>

$$S = \sqrt{2} \int ds \sqrt{E - V} , \quad (1)$$

with

$$\begin{aligned} ds^2 &= \sum_I \frac{m_I \{dq^{\mu I} - da^\mu - \{db \times q^I\}^\mu\} \{dq_{\mu I} - da_\mu - \{db \times q^I\}_\mu\}}{2} \\ &= \frac{m_I \delta_{\mu\nu} \delta_{IJ} \{dq^{\mu I} - da^\mu - \{db \times q^I\}^\mu\} \{dq^{\nu J} - da^\nu - \{db \times q^J\}^\nu\}}{2} \end{aligned} \quad (2)$$

(using the Einstein summation convention in the second expression, and where  $a^\mu$  and  $b^\mu$  are translational and rotational auxiliary variables).

2) By inclusion of these auxiliary corrections, RPM's are indirectly rendered *configurationally relational* [1, 11, 12, 10, 13]. This can be thought of as some group  $G$  of transformations that act on the theory's configuration space  $Q$  being taken to be physically meaningless above, this is the 'Euclidean group' of translations and rotations.

Note 1) In the relative Jacobi coordinates presentation (defined in Sec 2 and essentially the same form and more convenient for further discussions here), the kinetic line element is

$$ds^2 = \sum_i \frac{\mu_i \{dR^{i\mu} - \{db \times R^i\}^\mu\} \{dR_{\mu}^i - \{db \times R^i\}_\mu\}}{2} = \frac{\mu_i \delta_{\mu\nu} \delta_{ij} \{dR^{i\mu} - \{db \times R^i\}^\mu\} \{dR^{j\nu} - \{db \times R^j\}^\nu\}}{2} . \quad (3)$$

Note 2) The above indirectly formulated case with (3) in particular, is closely analogous to Baierlein–Sharp–Wheeler-type actions [14, 11, 10, 7] for GR in geometrodynamical form,

$$S = \int_{\Sigma} \int_{\Sigma} \sqrt{h} ds_{GR} \sqrt{R - 2\Lambda} , \quad \text{where } ds_{GR}^2 = M^{\mu\nu\rho\sigma} \{dh_{\mu\nu} - \mathcal{L}_{dF} h_{\mu\nu}\} \{dh_{\rho\sigma} - \mathcal{L}_{dF} h_{\rho\sigma}\} \quad (4)$$

is the GR kinetic line element. Thus, GR can also be cast in relational form in this paper's sense. [Note that (1) is equivalent to the Euler–Lagrange action and (4) is equivalent to the ADM Lagrangian action: see e.g. [7].] Next, I explain consequences of temporal relationalism and indirectly-implementing configurational in terms of constraints. Parametrization irrelevance then directly produces primary constraints quadratic in the momenta. In the GR case, the constraint arising thus is the super-Hamiltonian constraint,

$$\mathcal{H} := N_{\mu\nu\rho\sigma} \pi^{\mu\nu} \pi^{\rho\sigma} / \sqrt{h} - \sqrt{h} \{R - 2\Lambda\} = 0 , \quad (5)$$

whilst for RPM's it gives an energy constraint,

$$H := \sum_{i=1}^n P_i^2 / 2\mu_i + V = E . \quad (6)$$

Configurational relationalism can be approached by [1, 2] indirect means that amount to working on the principal bundle  $P(Q, G)$ . In this setting, linear constraints arise from varying with respect to the  $G$ -generators. In the case of GR, such a variation with respect to the generators of the 3-diffeomorphisms gives the super-momentum constraint

$$\mathcal{L}_\mu := -2D_\nu \pi^\nu_\mu = 0 , \quad (7)$$

<sup>1</sup>The notation for this series of papers is as follows. I use upper-case Latin letters as particle label indices running from 1 to  $N$  for particle positions, lower-case Latin letters as relative position variables indices running from 1 to  $n = N - 1$ , barred lower-case Latin letters as indices running from 1 to  $n - 1$  and lower-case Greek letters as spatial indices. I term 1-d RPM's *N-stop metrolands* from their configurations looking like public transport line maps, and 2-d ones *N-a-gonlands*, the first nontrivial two of which are *triangleland* and *quadrilateralland*. I use  $A$  indices for shape coordinates (running from 1 to 6 for quadrilateralland and from 1 to 3 for triangleland), and  $a$  indices for a subset of these running from 1 to 5 for quadrilateralland.  $\Sigma$  is a spatial manifold of fixed topology, taken to be compact without boundary for simplicity.  $h_{\mu\nu}$  is the spatial 3-metric, with determinant  $h$ , undensitized supermetric  $N_{\mu\nu\rho\sigma} = h_{\mu\rho} h_{\nu\sigma} - h_{\mu\nu} h_{\rho\sigma} / 2$ , conjugate momentum  $\pi^{\mu\nu}$ , covariant derivative  $D_\mu$  and Ricci scalar  $R$ .  $dF$  is the differential of the frame' counterpart of the shift in the relational formulation [10].  $N$  is the inverse of  $M$ .  $E$  is the total energy,  $V$  is the potential energy, and  $R^i$  are relative Jacobi interparticle (cluster) coordinate vectors (see Figure 1 in Sec 2 for further specification of what these are), with associated masses  $\mu_i$  and conjugate momenta  $P_i$ . Round brackets denote function dependence, square brackets denote functional dependence,  $( ; )$  denotes mixed function–functional dependence and  $[ ]$  denotes a portmanteau of function dependence in the finite case and functional dependence in the field-theoretic case.

whilst for RPM's variation with respect to the generators of the rotations gives that the total angular momentum for the model universe as a whole is constrained to be zero,

$$\mathbf{L} := \sum_{i=1}^n \mathbf{R}^i \times \mathbf{P}_i = 0. \quad (8)$$

In the case of pure-shape RPM, there is a  $+cdq^I$  in each bracket of (2) and a  $+cd\mathbf{R}^i$  in each bracket of (3). [It is also conventional for (2) and (3) to carry an overall  $1/I$  factor, making these homogeneous of degree zero as is most obviously appropriate for a pure-shape theory; in that convention, then, the potential term must be homogeneous of degree zero and the energy equation picks up an  $I$ -factor on its kinetic term]. This does not affect the outcome of  $\mathbf{b}$ -variation, which is pure-shape, whilst variation with respect to  $c$  gives that the total dilational momentum of the model universe is constrained to be zero,

$$\mathbf{D} := \sum_{i=1}^n \mathbf{R}^i \cdot \mathbf{P}_i = 0. \quad (9)$$

The momentum-velocity relations and the equations of motion for the RPM's then feature the combination  $[3, 15, 7]$   $\sqrt{(E - V)}\partial/\partial\mathbf{s} := d/dt^{\text{em}} := *$ , for  $t^{\text{em}}$  the *emergent time* of the relational approach. Such an emergent time amounts to a recovery of Newtonian, proper and cosmic time in various different contexts, as well as coinciding with the semiclassical notion of emergent time mentioned below. We then denote this joint notion of emergent time by  $t^{\text{em}}$ . The momentum-velocity relations are then  $\mathbf{P}_{i\mu} = \sqrt{(E - V)}/T \mu_i \{*\mathbf{R}_{i\mu} - \{\mathbf{b} \times \mathbf{R}_i\}_\mu\} := \mu_i \dot{\mathbf{R}}_{i\mu}$ . See [13] for the equations of motion, which additionally preserve the above constraints.

Next, elimination of the G-generator variables from the Lagrangian form of these constraints sends one to the desired quotient space  $\mathbf{Q}/\mathbf{G}$ . In 1- and 2- $d$ , these RPM's can be obtained on alternative foundations by working directly on  $\mathbf{Q}/\mathbf{G}$  [12, 13, 7] (see Sec 3 to 5 for further details). This involves applying a 'Jacobi-Synge' construction of the natural mechanics associated with a geometry, to Kendall's shape spaces [16] for pure-shape RPM's or to the cones over these [13] for the scaled RPM's. Direct implementation of configurational relationalism amounts to gauge-invariant variables (see Paper II for more on these/the significance of having a full set of these).

RPM's are further motivated as useful toy models (see [17, 3, 4, 11, 2, 18, 19, 20, 21, 15, 22, 23, 24, 10, 25, 13, 26, 8] and especially [7]) of yet further features of GR cast in geometrodynamical form. RPM's are similar in complexity and in number of parallels with GR to minisuperspace [27] and to  $2 + 1$  GR [28] are, though each such model differs in how it resembles GR, so that such models offer complementary perspectives and insights.<sup>2</sup> Thus RPM's are to be expected to be comparably useful to how minisuperspace and  $2 + 1$  gravity have been. In particular,

- I) The analogies with GR continue at the level of configuration spaces (see 7) .
- II) The analogies include the frozen formalism facet of the Problem of Time and various other of its facets (see the Conclusion for details), as well as a number of problem of time strategies having meaningful counterparts for RPM'.
- III) The nontrivial linear constraints parallel the GR momentum constraint (and are absent for minisuperspace), which is the cause of a number of further difficulties in various strategies to the Problem of Time (some are named below, see the Conclusion again for the meanings of these names; Paper IV of this series is dedicated to modelling many of these for the relational quadrilateral).

By parallels I-III), RPMs are appropriate as toy models for a large number of Problem of Time strategies [see below for more details and references]. Other useful applications of RPM's not covered by minisuperspace include the following.

- IV) RPM's are useful for the qualitative study of the quantum-cosmological origin of structure formation/ inhomogeneity. (Scaled RPM's are a tightly analogous, simpler version of Halliwell and Hawking's [30] model for this; moreover scalefree RPM's such as this paper's occurs as a subproblem within scaled RPM's, corresponding to the light fast modes/inhomogeneities.) Relatedly, RPM's are likewise useful for the study of correlations between localized subsystems of a given instant. RPM's also allow for a qualitative study of notions of uniformity/of maximally or highly uniform states in Classical and Quantum Cosmology, which are held to be conceptually important notions in these subjects.

## 1.2 Outline of the Problem of Time

RPM's energy constraint parallels the GR super-Hamiltonian constraint  $\mathcal{H}$  in leading to the frozen formalism facet of the Problem of Time [31, 17, 29, 19, 5, 32, 7]. This notorious problem occurs because 'time' takes a different meaning in each of GR and ordinary quantum theory. This incompatibility underscores a number of problems with trying to replace these two branches with a single framework in situations in which the premises of both apply, such as in black holes or the very early universe. One facet of the Problem of Time that then appears in attempting canonical quantization of GR is due to  $\mathcal{H}$  being quadratic but not linear in the momenta, which feature and consequence are shared by  $\mathbf{H}$ . Then elevating  $\mathcal{H}$  to a quantum equation produces a stationary i.e timeless or frozen wave equation: the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = 0, \quad (10)$$

instead of ordinary QM's time-dependent one,

$$i\hbar\partial\Psi/\partial t = \hat{H}\Psi \quad (11)$$

(where we use  $\Psi$  for the wavefunction of the universe,  $H$  to denote a Hamiltonian and  $t$  for absolute Newtonian time).

<sup>2</sup>E.g. [17, 29, 19] use many such toy models in discussing the Problem of Time in Quantum Gravity and other foundational issues.

**Problem of Observables** [33, 5, 32]. Here, one tries to construct a sufficient set of observables for the physics of one’s model, which are then involved in the model’s notion of evolution. Addressing this depends on how observables are *defined*, e.g. whether one requires observables to commute with all constraints or just with those that are linear in the momenta (see Paper II for more). See [17, 29, 32] for other facets of the Problem of Time. Some of the strategies toward resolving the Problem of Time in Quantum Gravity that can be modelled by RPM’s are as follows [7].

A) Perhaps one has slow heavy ‘ $h$ ’ variables that provide an approximate timestandard with respect to which the other fast light ‘ $l$ ’ degrees of freedom evolve [34, 30, 17, 19]. In the Halliwell–Hawking [30] scheme for GR Quantum Cosmology,  $h$  is scale (and homogeneous matter modes) and  $l$  are small inhomogeneities. Thus the scale–shape split of scaled RPM’s afford a tighter parallel of this [23, 35, 36] than pure-shape RPM’s. The semiclassical approach involves firstly making the Born–Oppenheimer ansatz  $\Psi(h, l) = \psi(h)|\chi(h, l)\rangle$  and the WKB ansatz  $\psi(h) = \exp(iW(h)/\hbar)$ . Secondly, one forms the  $h$ -equation ( $\langle\chi|\hat{H}\Psi = 0$  for RPM’s), which, under a number of simplifications, yields a Hamilton–Jacobi<sup>3</sup> equation

$$\{\partial W/\partial h\}^2 = 2\{E - V(h)\} \quad (12)$$

for  $V(h)$  the  $h$ -part of the potential. Thirdly, one way of solving this is for an approximate emergent semiclassical time  $t^{\text{em}} = t^{\text{em}}(h)$ . Next, the  $l$ -equation  $\{1 - |\chi\rangle\langle\chi|\}\hat{H}\Psi = 0$  can be recast (modulo further approximations) into an emergent-time-dependent Schrödinger equation for the  $l$  degrees of freedom

$$i\hbar\partial|\chi\rangle/\partial t^{\text{em}} = \hat{H}_l|\chi\rangle. \quad (13)$$

(Here the left-hand side arises from the cross-term  $\partial_h|\chi\rangle\partial_h\psi$  and  $\hat{H}_l$  is the remaining surviving piece of  $\hat{H}$ ). Note that the working leading to such a time-dependent wave equation ceases to work in the absense of making the WKB ansatz and approximation, which, additionally, in the quantum-cosmological context, is not known to be a particularly strongly supported ansatz and approximation to make.

B) A number of approaches take timelessness at face value. One considers only questions about the universe ‘being’, rather than ‘becoming’, a certain way. This has at least some practical limitations, but can address some questions of interest. As a first example, the *naïve Schrödinger interpretation* [37] concerns the ‘being’ probabilities for universe properties such as: what is the probability that the universe is large? Flat? Isotropic? Homogeneous? One obtains these via consideration of the probability that the universe belongs to region  $R$  of the configuration space that corresponds to a quantification of a particular such property,

$$\text{Prob}(R) \propto \int_R |\Psi|^2 d\Omega, \quad (14)$$

for  $d\Omega$  the configuration space volume element. As a second example, the *conditional probabilities interpretation* [38] goes further by addressing conditioned questions of ‘being’ such as ‘what is the probability that the universe is flat given that it is isotropic’? As a final example, *records theory* [38, 39, 4, 40, 24] involves localized subconfigurations of a single instant. More concretely, it concerns whether these contain useable information, are correlated to each other, and a semblance of dynamics or history arises from this. This requires notions of localization in space and in configuration space as well as notions of information. RPM’s are superior to minisuperspace for such a study as, firstly, they have a notion of localization in space. Secondly, they have more options for well-characterized localization in configuration space (i.e. of ‘distance between two shapes’ [7]) through their kinetic terms possessing positive-definite metrics.

C) Perhaps instead it is the histories that are primary (*histories theory* [39, 41]).

Combining A) to C) (for which RPM’s are well-suited) is a particularly interesting prospect [42], along the following lines (see also [39, 40, 43, 7] for further development of this). There is a records theory within histories theory. Histories decohering is one possible way of obtaining a semiclassical regime in the first place, i.e. finding an underlying reason for the crucial WKB assumption without which the semiclassical approach does not work. What the records are will answer the also-elusive question of which degrees of freedom decohere which others in Quantum Cosmology.

D) Observables-based approaches [5] can also be studied in the RPM arena, as can be quantum cosmologically aligned hidden time approaches for *scaled* RPM’s – some of this is covered in Paper II and some of it in Paper IV.

### 1.3 Unlocking RPM’s: from the simplest through to quadrilateralland

Key 1 Jacobi coordinates work [20, 44] for all RPM’s regardless of dimension  $d$  or particle number  $N$ . It turns out to be advantageous to treat the pure-shape case first.

<sup>3</sup>For simplicity, I present this in the case of 1  $h$  degree of freedom and with no linear constraints.

Key 2 topology is only simple for 3 series, and only two of these remain metrically simple. This is due to Kendall and Casson (see e.g. [16]) and slightly repackaged for use in the physical modelling of relational whole-universe models in [7]. The two metrically simple series are the  $N$ -stop metrolands (configuration space  $\mathbb{S}^{N-2}$ ) and the  $N$ -a-gonlands (configuration space  $\mathbb{CP}^{N-2}$ ). These have standard spherical and Fubini–Study metrics. The former appears most naturally in Beltrami coordinates, but then

Key 3[ $N$ -stop] is identifying it and simplifying things is done by passage to (ultra)spherical coordinates. The simplifications include that these cast the metric in diagonal form.

Key 3[ $N$ -a-gon] is use of inhomogeneous coordinates; these do not however get close to diagonal.

Key 4 is use of coning. For  $N$ -stop metrolands, this gives the configuration space  $C(\mathbb{S}^{N-2}) = \mathbb{R}^{N-1}$ , whilst for  $N$ -a-gonlands it gives the configuration space  $C(\mathbb{CP}^{N-2})$  which does not in general simplify.

Key 5[ $N$ -stop] is that the Jacobi magnitudes themselves will do as coordinates for this  $\mathbb{R}^{N-1}$ .

Key 6[ $N$ -stop] is that these also then serve as subsystem-describing coordinates.

Modelling Feature 1) Nontrivial clustering/inhomogeneity/structure (a midisuperspace-like feature).

Modelling Feature 2) For  $N \geq 4$  particle models, one has relationally nontrivial non-overlapping subsystems and hierarchies of nontrivial subsystems (see the Conclusion for applications of these).

$N$ -stop metrolands have been used as toy models for the following Problem of Time approaches: the Naïve Schrödinger Interpretation [25, 22], the semiclassical approach [15, 35, 7], histories theory [7] and internal time [21, 15, 7].

N.B. 2- $d$  suffices for almost all the analogies with GR to hold whilst still keeping the mathematics manageable.

Modelling Feature 3) Nontrivial linear constraints (another midisuperspace-like feature).

Modelling Features 1) and 3) together render this suitable as a qualitative toy model of Halliwell and Hawking’s [30] quantum cosmological origin of structure formation in the universe).

Triangleland was covered in [6, 45, 46, 47]. See also [49, 50, 51, 52] for in some ways similar accounts of spaces of triangles. [53] has the first use of the tessellation presentation I subsequently used in [46, 47].

The first nontrivial  $N$ -a-gonland – triangleland – has the good fortune that  $\mathbb{CP}^1 = \mathbb{S}^2$ .

Key 3b[ $\triangle$ ] The spherical coordinates again serve to diagonalize the configuration space line element, albeit these now have a new interpretation (see Sec 5.3).

Key 5[ $\triangle$ ] Dragt coordinates [54]: the surrounding Cartesian coordinates, rather less straightforwardly realized as there are now four rather than just three obvious Jacobi vector quantities to use, so they need repackaging, and lucidly interpreted as shape quantities aniso, area, ellip and furthermore enter the kinematical quantization...

Key 5b[ $\triangle$ ] is that these include a democracy invariant (the area of the triangle).

Key 6[ $\triangle$ ] is now attained by the parabolic coordinates. (These are 2 partial moments of inertia and the relative angle between the associated Jacobi vectors).

Key 7[ $\triangle$ ] is tessellation by the physical interpretation, as documented for the triangle at the topological level by Montgomery [55, 51], and at the metric level by Kendall [53]. This was applied to whole-universe physics in [46], as well as extended to other RPM’s in [25] and the present Paper.

Then of course, the sphere has a lot of standard geometry and Methods of Mathematical Physics available for it [6, 46]; there is almost as much for the  $k$ -sphere.

Triangleland has lucid interpretation of configuration space regions [46], with Naïve Schrödinger Interpretation [46, 47], and Halliwell combined approach applications (see [26] for these applications and [40, 42, 43] for Halliwell’s work itself). The latter are supported also by RPM’s having a Problem of Observables resolution [26] (of which the shapes and shape momenta are a lucid account, see Paper II) and a histories theory setup [7]. Triangleland is also tractable from Records Theory [24, 7], Semiclassical approach [35], Histories theory [7] and internal time [7] perspectives; I consider Halliwell’s combination of the first three of these in [26]. Triangleland possesses a notion of uniformity [46, 7]. [35] demonstrates accessibility of qualitative checks of the Halliwell–Hawking semiclassical inhomogeneous perturbations about minisuperspace approach to the origin of structure in the universe.

The triangle does not however have Modelling Feature 2). It is Quadrilateralland that first unifies these three features. I bill quadrilateralland as similar in complexity and in number and depth of applications to the non-diagonal Bianchi IX models. For quadrilateralland, unlike with all previous RPM’s, Molecular Physics no longer provides kinematics and analogies. Nevertheless, we still manage to find counterparts of all the Keys in the present series of papers.

Key 3[ $\square$ ] The  $N$ -a-gons all possess complex projective space mathematics (triangleland atypically simplifies via  $\mathbb{CP}^1 = \mathbb{S}^2$ , whereas quadrilateralland is much more mathematically typical for an  $N$ -a-gonland, the first typical such). Quadrilateralland extends triangleland for some purposes, for all that many contexts other than this paper (Celestial Mechanics, Molecular Physics, Nuclear Physics, simplex constructions, spin foams) pass instead to the tetrahedron.  $\mathbb{CP}^2$  geometry is in this paper. The associated Methods of Mathematical Physics is considered in Paper III (this is still within standard: Jacobi polynomials and Wigner D-functions). Fubini–Study metric in inhomogeneous coordinates for 2- $d$  RPM’s [12, 16]. These are analogues of the Beltrami coordinates.

Key 4[ $\square$ ] continues to hold as a straightforward extension, though, unlike for  $N$ -stop metroland and triangleland, the ensuing spaces are not as well-known.

Key 5[□] Parallel the Dragt-type coordinates [54] that were so useful for triangle-land [46, 47] (and are closely related to the Hopf map). ii) The clustering-independent (‘democratically invariant’ [56, 57, 52]) properties for this. iii) This leads to a quantifier of uniformity for model-universe configurations. The present paper is the first instance in RPM literature involving detailed treatment of indistinguishable particles as well as the first paper on submanifolds within  $\mathbb{CP}^2$  interpreted in quadrilateral-land terms. These two papers, and a third concerning the interpretation of  $\mathbb{CP}^2$ ’s  $SU(3)$  of conserved quantities in quadrilateral-land terms [58], are important prerequisites for the subsequent study of the classical and quantum mechanics of quadrilateral-land [59]. The eventual application to the Problem of Time in Quantum Gravity is mostly in Paper IV [60].

Key 3b[□] parallel the sphericals: intrinsic and ‘block-minimal’ (i.e. as diagonal as possible).

Key 6[□] Parallel the parabolic coordinates and physically correspond to subsystem split.

Note: for scale-free, Dragt coordinates and parabolic coordinates have 1 redundancy each.

Key 7[□] is less available due to 4- $d$  manifolds being harder to visualize, but one can still ask about the meaningful submanifolds within.

Thus I pose what are useful coordinate systems as questions, with 4-stop/triangle-land counterparts motivating various kinds of structures. Given the above Keys, I unlocked the classical dynamics, QM and Problem of Time aspects of the triangle in, respectively, [48, 6, 47], [45, 46, 47] and [46, 47, 35, 7, 26]. Then Paper II covers the classical mechanics, Paper III covers the QM a(including the extension to the scaled case) and finally Paper IV provides the Problem of Time strategy studies in the arena of the relational quadrilateral.

## 1.4 Further motivation for $\mathbb{CP}^2$ study itself

Key 9 is knowledge of the isometry groups and that these turn out to be mathematically simple. They are  $SO(N-1)$  for  $N$ -stop metrolands and  $SU(N-1)/\mathbb{Z}_{N-1}$  for  $N$ -a-gonlands, which is just  $SO(3)$  for triangle-land and is  $SU(3)/\mathbb{Z}_3$  for quadrilateral-land. See Paper II for more. I mention this here as regards +motivation.  $\text{Isom}(\mathbb{CP}^2) = SU(3)/\mathbb{Z}_3$  which is sufficiently similar to  $SU(3)$  to give the same representation theory and mathematical form of conserved quantities as in the idealized flavour  $SU(3)$  or the colour  $SU(3)$  of Particle Physics [61]. [One can argue further for colour involving this quotient rather than  $SU(3)$  itself.] In the  $\mathbb{CP}^2$  context, the  $SU(3)$  is nonlinearly realized; this has been studied by MacFarlane [62, 63] (some results of which are already in [64]; also see [65]). Paper II then provides the quadrilateral-land interpretation of  $\mathbb{CP}^2$ ’s  $SU(3)$  of conserved quantities. Other  $\mathbb{CP}^2$  interpretations include the following.

A) qutrits in Quantum Information Theory [66]; these can also be interpreted in terms of quadrilaterals and qunits in terms of  $N$ -a-gons for  $N = n + 1$ . These are motivated by there being much more information storage in a system based on qunits rather than on qubits.

B)  $SU(3)$  corresponding to  $\mathbb{CP}^2$  also occurs in the theory of skyrmions and in nonlinear sigma models; these are good toy models [67] for some aspects of Yang–Mills theory – these models possess instantons,  $\theta$ -vacua and topological quantum numbers.  $\mathbb{CP}^2$  is here present in its quotient form as the targetspace [68].

C) In the study of gravitational instantons,  $\mathbb{CP}^2$  is Einstein, and hence a solution of the Euclidean-signature version of GR [69]. (This rests on  $\mathbb{CP}^2$  being Einstein and instantons in any case being Euclideanizations so that the ++++ signature does not get in the way of this application) [70].

D) Finally, complex projective spaces play a prominent role in twistor theory (e.g. planes in projective twistor space are given by  $\mathbb{CP}^2$ ’s [71]).

## 1.5 Outline of the rest of this paper

In Sec 2 I provide coordinate systems and types of configuration space that are useful in the study of the general relational particle mechanics models; this covers Key 1. I then consider details of the configuration spaces in Sec 3, covering Keys 2, 4 and parts of Key 3: nontrivial and mathematically well-known spaces: spheres and complex projective spaces, and cones over these spaces. In Sec 4, I cover mirror-image-identified and indistinguishable-particle variants. In Sec 5 I then consider the specific further Keys for 4-stop metroland and triangle-land, alongside further coordinatizations useful in these specific cases. I summarize configuration space information so far in Sec 6, including comparison with the situation in GR. I consider collapses of Jacobi vectors in Sec 7; this is a start on notions of merger and particular submanifolds for quadrilateral-land, which topic goes toward resolving Key 7[□]. I then start with quadrilateral-land extensions of all the above, beginning by considering democracy transformations and shape coordinates in Sec 8 (these resolve Key 5[□]). I then consider various sets of coordinates related to these in Sec 9 (these resolve Key 6[□]), alongside the distinct intrinsic Gibbons-Pope-type coordinates that resolve Key 3b[□]. I then apply these to the ‘equator and hemispheres’ problem in Sec 10, as well as to further characterizing submanifolds of collisions and mergers. I consider notions of uniformity for quadrilateral-land in Sec 11. I conclude in Sec 12 by listing the answers to the Keys, providing also some comments on the higher- $N$  extension of the present paper. I also provide a discussion of regions of configuration space, which constitute Key 8 and are to be applied to the Naïve Schrödinger Interpretation, semiclassical and Halliwell approaches to Problem of Time; for instance, I consider approximate collinearity as a thickening of the submanifold of collinearity. (Some of these things have appeared before in the preprints [72, 73]; however those preprints are not for publication, the present article covering both of those, and more.)

## 2 Positions, relative positions and Jacobi vectors

Consider  $N$  particle-position vectors  $\mathbf{q}^I$ . We are really dealing with constellations and not figures, which is first a significant distinction for this paper's case of the quadrilateral: 'joining the dots' is no longer unique and which way they are joined is not particularly meaningful.

$N$  particles in dimension  $d$  has an incipient Cartesian *configuration space*  $Q(N, d) = \mathbb{R}^{Nd}$ . This is for now based on a Cartesian absolute space  $A(d) = \mathbb{R}^d$ . There is then for now absolute position, absolute rotation and absolute scale.

Rendering absolute position irrelevant (e.g. by passing from  $\mathbf{q}^I$  to any sort of relative coordinates) leaves one on the configuration space *relative space*,  $R = \mathbb{R}^{2n}$ , for  $n = N - 1$ . Relative Lagrangian coordinates are the most obvious for this: a basis set of  $n$  of the  $\mathbf{r}^{IJ} = \mathbf{r}^J - \mathbf{r}^I$ . The kinetic line element is unfortunately not diagonal in these.

This is resolved by using instead  $N - 1$  relative Jacobi vectors [44],  $\mathbf{R}^i$ . These are combinations of relative position vectors between particles that form in general inter-particle cluster vectors. Relative Jacobi coordinates have associated particle cluster masses  $\mu_e$ . In fact, it is tidier to use *mass-weighted* relative Jacobi coordinates  $\rho^i = \sqrt{\mu_i} \mathbf{R}^i$  (Fig 1). The squares of the magnitudes of these are the partial moments of inertia  $I^i = \mu_i |\mathbf{R}^i|^2$ . I also denote  $|\rho^i|$  by  $\rho^i$ ,  $I^i/I$  by  $N^i$  for  $I$  the moment of inertia of the system, and  $\rho^i/\rho$  by  $\mathbf{n}^i$  for  $\rho = \sqrt{I}$  the *configuration space radius* (called *hyperradius* in the Molecular Physics literature).

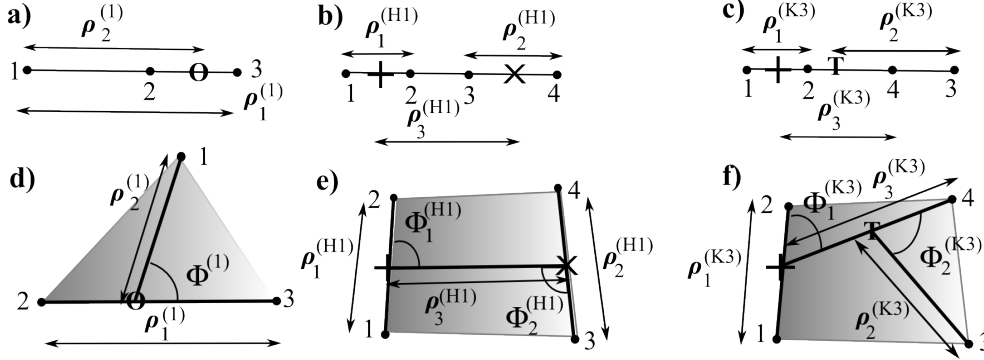


Figure 1: a) 3-stop metroland's Jacobi coordinates. b) 4-stop metroland's Jacobi H-coordinates. c) 4-stop metroland's Jacobi K-coordinates. These are 'squashed' versions of d), c) and e) respectively, for which the tree names T, H and K are self-evident. d) Triangleland's Jacobi coordinates. I term the two separations of the Jacobi triangle as the *base* and the *median*. e) Quadrilateralland in Jacobi H-coordinates. I term the three separations of the Jacobi H tree as the two *posts* separated by the *crossbar*. f) Quadrilateralland in Jacobi K-coordinates. I term the three separations of the Jacobi K tree as the *back*, the *seat* and the *leg* (viewing it as the cross-section of a chair). O, +, × and T denote COM(23), COM(12), COM(34) and COM(124) respectively, where COM(ab) is the centre of mass of particles a and b. I write specific coordinate components' indices downstairs for ease of presentation.

For overall compactness of presentation, I also include here the definitions of the following relative angles that are used in subsequent Sections. For the triangle,  $\Phi^{(a)}$  is the 'Swiss army knife' angle  $\arccos(\rho_1^{(a)} \cdot \rho_3^{(a)} / \rho_1^{(a)} \rho_3^{(a)})$ . For Jacobi H-coordinates [with  $\Phi_1^{(Hb)}$  and  $\Phi_2^{(Hb)}$  are the 'Swiss army knife' angles  $\arccos(\rho_1^{(Hb)} \cdot \rho_3^{(Hb)} / \rho_1^{(Hb)} \rho_3^{(Hb)})$  and  $\arccos(\rho_2^{(Hb)} \cdot \rho_3^{(Hb)} / \rho_2^{(Hb)} \rho_3^{(Hb)})$ , whilst for Jacobi K-coordinates,  $\Phi_1^{(Ka)}$  and  $\Phi_2^{(Ka)}$  are the 'Swiss army knife' angles  $\arccos(\rho_1^{(Ka)} \cdot \rho_3^{(Ka)} / \rho_1^{(Ka)} \rho_3^{(Ka)})$  and  $\arccos(\rho_2^{(Ka)} \cdot \rho_3^{(Ka)} / \rho_2^{(Ka)} \rho_3^{(Ka)})$ . I note that there is a third relative angle for each tree's representation of quadrilateralland - that between the inclinations of the posts or of the back and the leg; whilst these are in each case not independent, it is sometimes useful to talk about them, and to I term these *post angles*. These carry the implication of involving extrapolating a Jacobi separation in order to be defined locally.

**Jacobi trees.** For 3 particles, one particular choice of mass-weighted relative Jacobi coordinates are as indicated in Fig 1.d). This is a T-shaped tree on the Jacobi clustering structure of the constellation. For 4 particles, there are then (up to permutation) two possible trees on the Jacobi clustering structure: the H and K of Fig 1.e) and f). See e.g. [57] for further trees. (Graph-theoretically, *trees* are 'loopless and connected', as is familiar in theoretical physics from the use of 'tree amplitudes'; moreover, these trees are also *irreducible*: no vertices of degree 2. N.B. the constellation does not in general constitute the vertices due to the benefits of diagonality, attained from the Jacobi clustering structure. This means that a number of the vertices are centres of mass of clusters rather than point particles, hence 'Jacobi tree'.)

**Cluster notation** Each of the T's is then labelled according to its clustering structure: I use {a, bc} read left-to-right in 1- $d$  and anticlockwise in  $\geq 2$   $d$ , which I abbreviate by (a). I use {a...c} to denote a cluster composed of particles a, ... c, ordered left to right in 1- $d$  and anticlockwise in 2- $d$ ; I take these to be distinct from their right to left and clockwise counterparts i.e. I consider plain configurations. I insert commas and brackets to indicate a clustering, i.e. a partition into clusters. These notations also cover collisions, in which constituent clusters collapse to a point. I use (Hb) as shorthand for {ab, cd} i.e. the clustering (partition into subclusters) into two pairs {ab} and {cd}, and (Ka) as shorthand for {{cd, b}, a} i.e. the clustering into a single particle a and a triple {cd, b} which is itself partitioned into a pair cd and a single particle b. In each case, a, b, c, d form a cycle. In this Sec and the next, I take clockwise and anticlockwise labelled triangles and quadrilaterals to be distinct, i.e. I make the plain rather than mirror-image-identified choice of set of shapes. I also consider just distinguishable particles. See Sec 4 for otherwise on these two counts. I also assume equal masses for simplicity. Later references to H and K coordinates refer explicitly to the (H1) and (K3) cases depicted above; I drop these labels to simplify the notation.

Note: I emphasize that it is the constellation and not the tree, cluster-label or the shape made by joining the dots that is relational; moreover, we shall see that the choice of tree and of cluster label can have particular *contextual* meaning as regards the regime of study or the propositions being addressed. E.g. if the three particles are the Sun, Earth and Moon, then there is particular significance to having the apex be the Sun (as the Earth and Moon base pair are far more localized) or the Moon (as it is considerably lighter and therefore amenable to treatment as a perturbation). E.g. H-coordinates are particularly suited to the study of two pairs of binary stars/diatomics/H atoms, whilst K-coordinates are suited to a binary/diatomic/H atom alongside two single bodies/to emphasizing a triple particle subsystem within.

### 3 Topological and metric structure of configuration space

If one quotients out the rotations also, one is on *relational space*  $\mathcal{R}(N, d)$ . If one furthermore quotients out the scalings, one is on *shape space*  $\mathcal{S}(N, d)$ . If one quotients out the scalings but not the rotations, one is on *preshape space* [16]  $\mathcal{P}(N, d)$ . Key 2 for  $N$  particles in 1- $d$ , preshape space is  $\mathbb{S}^{N-2}$  (or some piece thereof, see Secs 4, 5) and this coincides with shape space. For  $N$  particles in 2- $d$ , preshape space is  $\mathbb{S}^{2n-1}$  and shape space is  $\mathbb{CP}^{N-2}$  (provided that the plain choice of set of shapes is made). The 3-particle case of this is, moreover, special, by  $\mathbb{CP}^1 = \mathbb{S}^2$ .  $\mathbb{CP}^{n-1}$  involves  $n$  lines, whilst  $n$  lines can be used to form whichever Jacobi tree for an  $N$ -a-gon. This is a lucid insight as to why  $\mathbb{CP}^{n-1}$  is representable as the space of all  $N$ -a-gons.

Key 4 (topological part) also, relational space  $\mathcal{R}(N, d)$  is equal to [13] *the cone* over shape space, denoted by  $\mathcal{C}(\mathcal{S}(N, d))$ . [At the topological level, for  $\mathcal{C}(X)$  to be a cone over some topological manifold  $X$ ,

$$\mathcal{C}(X) = X \times [0, \infty) / \sim, \quad (15)$$

where  $\sim$  means that all points of the form  $\{p \in X, 0 \in [0, \infty)\}$  are ‘squashed’ or identified to a single point termed the *cone point*, and denoted by 0. For what a cone further signifies at the level of Riemannian geometry, see below.] In particular,  $\mathcal{R}(N, 1) = \mathcal{C}(\mathbb{S}^{N-2}) = \mathbb{R}^n$  and  $\mathcal{R}(N, 2) = \mathcal{C}(\mathbb{CP}^{N-2})$  [among which additionally  $\mathcal{C}(\mathbb{CP}^1) = \mathcal{C}(\mathbb{S}^2) = \mathbb{R}^3$  at the topological level]. Thus, this paper’s quadrilateralland case is the first case with nontrivial complex-projective mathematics.

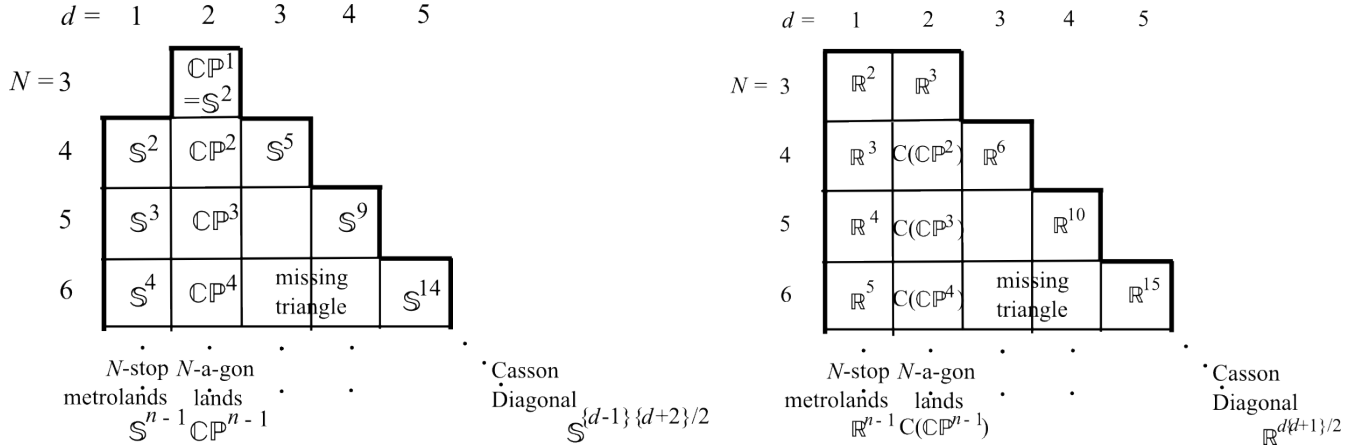


Figure 2: ‘Partial Periodic Table’ of the discernible a) shape spaces and b) relational spaces at the topological level. In each case, only the first two columns (rather than the ‘Casson diagonal’ [16]) continue to be straightforward at the metric level. These constitute Key 2 to unlocking RPM’s.

Key 3 for 1- $d$ , one has the usual  $\{N - 2\}$ -sphere metric on shape space,  $d^2 s_{\text{sph}}$ , with then the Euclidean metric on the corresponding relational space,  $d^2 s_{\text{Eucl}} = d\rho^2 + \rho^2 d^2 s_{\text{sph}}$ . For 2- $d$ , the kinetic metric on shape space is the natural Fubini–Study metric [16, 12],

$$ds_{\text{FS}}^2 = \{ \{1 + |Z|_c^2\} |dZ|_c^2 - |(Z, d\bar{Z})_c|^2 \} / \{1 + |Z|_c^2\}^2. \quad (16)$$

I explain the coordinates in use here as follows. Firstly,  $\{z^i\}$  are, mathematically, complex homogeneous coordinates for  $\mathbb{CP}^2$ ; I denote their polar form by  $z^i = \rho^i \exp(i\phi^i)$ . Physically, these contain 2 redundancies and their moduli are the magnitudes of the Jacobi vectors whilst their arguments are angles between the Jacobi vectors and an absolute axis. Next,  $\{Z^{\bar{p}}\}$  are, mathematically, complex inhomogeneous coordinates for  $\mathbb{CP}^2$ ; I denote their polar form by  $Z^{\bar{p}} = \mathcal{R}^{\bar{p}} \exp(i\Phi^{\bar{p}})$ . These are independent ratios of the  $z^i$ , and so, physically their magnitudes  $\mathcal{R}^{\bar{p}}$  are ratios of magnitudes of Jacobi vectors, and their arguments  $\Phi^{\bar{p}}$  are now angles between Jacobi vectors, which are entirely relational quantities. Also, I use  $|Z|_c^2 := \sum_{\bar{p}} |Z^{\bar{p}}|^2$ ,  $(, )_c$  for the corresponding inner product, overline to denote complex conjugate and  $||$  to denote complex modulus. Note that using the polar form for the  $Z^{\bar{p}}$ , the line element and the corresponding kinetic term can be cast in a real form. In 2- $d$ , this fails to cover Key 3b: does not come out in block-minimal form.

Key 4 (metric part) the kinetic metric on relational space in scale-shape split coordinates is then of the cone form

$$ds_{\mathcal{C}(\text{FS})}^2 = d\rho^2 + \rho^2 ds_{\text{FS}}^2 \quad (17)$$



[for  $ds^2$  is the line element of  $X$  itself and  $\rho$  a suitable ‘radial variable’<sup>4</sup> that parametrizes the  $[0, \infty)$ , which is the distance from the cone point; such ‘cone metrics’ are smooth everywhere except (possibly) at the troublesome cone point].

The action for RPM’s in relational form is then (1), with, in 1- or 2- $d$ ,  $ds^2 = ds_{\text{sph}}^2$  or  $ds_{\text{Eucl}}^2$ ,  $ds^2 = ds_{\text{FS}}^2$  or  $T_{\text{C(FS)}}$  built from the above selection of metrics (thus directly implementing configurational relationalism).

Note 1) In the spherical presentation of the triangleland case, coordinate ranges dictate that the radial variable is, rather,  $I$ . Also note that, whilst this cone is topologically  $\mathbb{R}^3$ , the metric it comes equipped with is *not* the flat metric (though it is exploitably *conformally flat* [6, 47]).

Note 2) RPM’s in 3- $d$  are much more intractable [16], whilst not enhancing much the geometrodynamical analogy.

## 4 Mirror-identified and indistinguishable-particle variants

This is a slight detour before the summary of configuration spaces involved; it is also relevant via some of these featuring as submanifolds within other of the configuration spaces

Because the indirect implementation of configurational relationalism below encodes continuous  $G$ . I furthermore need to consider here, at the preliminary level of choosing  $Q$  itself, whether physical irrelevance of the non-continuous reflection operation is to be an option or, indeed, obligatory. I.e. should  $Q$  be a space of *plain* configurations  $Q(N, d)$  or of *mirror-image-identified* configurations  $OQ(N, d) = Q(N, d)/\mathbb{Z}_2$  [the  $O$  stands for ‘orientation-identified’, meaning that the clockwise and anticlockwise versions are identified,  $O$  being a more pronounceable prefix than  $M$ ]. Mirror-image-identified configurations are always at least a mathematical option at this stage by treating all shapes and their mirror images as one and the same:  $\mathbb{R}^{Nd}/\mathbb{Z}_2$ . However, one shall see below that sometimes the model has symmetry enough that declaring overall rotations themselves to be irrelevant already includes such an identification. In this case one can not meaningfully opt out of including the reflection, so this fork degenerates to a single prong. I note that the  $\mathbb{CP}^{N-2}/\mathbb{Z}_2$  (with reflexive action) occurring here has mathematical analogues elsewhere in the Theoretical Physics literature: weighted projective spaces [74].

For indistinguishable particles, consider quotients by the permutation group on  $N$  objects,  $S_N$  or its even counterpart  $A_N$  that excludes the extra reflection generator present in  $S_N$ . (Alternatively, one could use the  $P < N$  versions of these if only a subset are distinguishable, or a product of such whose  $P$ ’s sum up to  $\leq N$  if the particles bunch up into internally indistinguishable but mutually distinguishable). This would be more Leibnizian (by identifying even more indiscernibles), as well as, more specifically, concordant with the nonexistence of meaningless distinguishing labels at the quantum level. (Though it is fine for particles to be distinguishable by differing in a *physical* property such as mass or charge or spin, which then actively enters the physical equations, and spin only arises at the quantum level.)

## 5 Configuration spaces and coordinate systems for 3- and 4-stop metroland and for triangleland

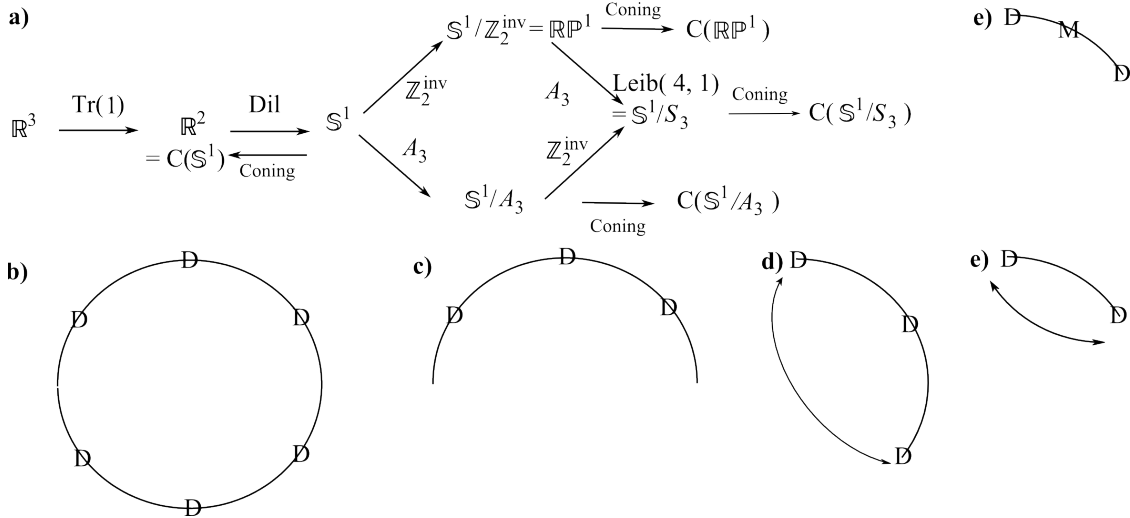


Figure 3: a) The 3-stop metroland circle (trivial as a shape space nontrivial as shape part of the scaled case. b) and c) are the tessellations for the distinguishable particles plain shape and mirror image identified cases respectively. d) and e) are then the indistinguishable particle counterparts of these; the double arrow indicates topological identification. f) The accompanying notion of merger for 3-stop metroland, physically corresponding to one particle lying upon the centre of mass of the other two, decorates each portion of the allowed configuration space as indicated.

<sup>4</sup>In the spherical presentation of the triangleland case, coordinate ranges dictate that the radial variable is, rather,  $I$ . Also note that, whilst this cone is topologically  $\mathbb{R}^3$ , the metric it comes equipped with is *not* the flat metric (though it is exploitably *conformally flat* [6, 47]).

## 5.1 3-stop metroland

3-stop metroland's shape space is the circular arc  $\mathbb{S}^1$ . It is decorated as in Fig 3a). As a dynamics of pure shape, this is trivial by degree of freedom count, but it is nontrivial as the shape part of the scaled 3-stop metroland theory. A useful coordinatization is by  $\varphi$  running from 0 to  $2\pi$ , whose physical meaning is

$$\varphi = \arctan(\rho_2/\rho_1) . \quad (18)$$

The corresponding RPM kinetic line element is

$$ds^2 = d\varphi^2/2 . \quad (19)$$

## 5.2 4-stop metroland

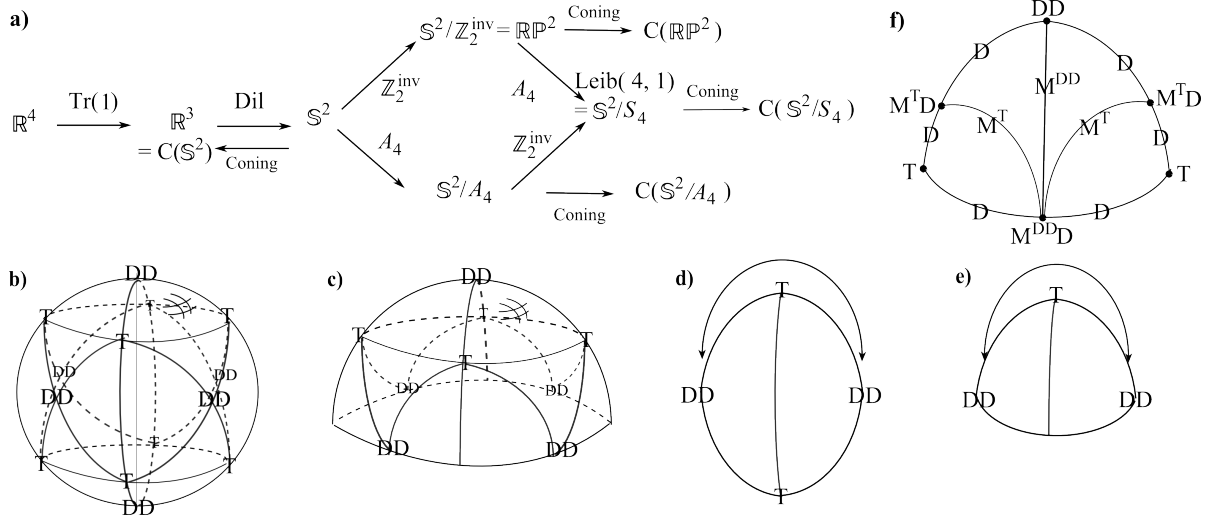


Figure 4: a) The sequence of configuration spaces for 4-stop metroland. b) and c) are the tessellations for the distinguishable particles plain shape and mirror image identified cases respectively. d) and e) are then the indistinguishable particle counterparts of these. f) illustrates the notions of merger new to 4-stop metroland within a single triangular face. There are  $M^T$  arcs (for which the fourth particle is at the centre of mass of the other three) and  $M^{DD}$  arcs (for which the centre of mass of 2 pairs of particles coincide), as well as  $M^T \cap D = M^T D$  and  $M^{DD} \cap M^T \cap D = M^* D$  points. These are also the types of merger present in quadrilateralland, though there of course their geometry as configuration space regions is more complicated.

4-stop metroland's shape space is the sphere  $\mathbb{S}^2$ , decorated as in Fig 2b). The configuration space here for distinguishable particles and in the plain shape case is the sphere, decorated with the physical interpretation of Figs 4c). All the lines in Fig 3c) to f) are lines of double collisions,  $D$ . The  $DD$  points are double double collisions (i.e. two separate double collisions) whilst the  $T$  points are triple collisions. At the level of configuration space geometry, the mirror-image-identification in space becomes inversion about the centre of the sphere. This gives rise to the real projective space,  $\mathbb{RP}^2$  [Fig 4d)]. Indistinguishability then involves quotienting out  $S_4$  (permutations of the particles, which is isomorphic to the cube or octahedron group acting on the  $DD$ 's or the  $T$ 's), as in Fig 3f). If there is no mirror-image identification, the quotienting out is rather by  $A_4$  (even permutations of the particles, isomorphic to the group of the cube or the octahedron excluding one reflection operation), as in Fig 3e).

A useful coordinatization is in terms of  $\theta$  running from 0 to  $\pi$  and  $\phi$  running from 0 to  $2\pi$ , whose physical meanings are, in Jacobi H-coordinates,

$$\theta = \arctan(\sqrt{\rho_1^2 + \rho_2^2}/\rho_3) , \quad \phi = \arctan(\rho_2/\rho_1) , \quad (20)$$

which are respectively a measure of the size of the universe's contents relative to the size of the whole model universe, and a measure of inhomogeneity among the contents of the universe (whether one of the constituent clusters is larger than the other one.) On the other hand, for Jacobi K-coordinates

$$\theta = \arctan(\sqrt{\rho_1^2 + \rho_2^2}/\rho_3) , \quad \phi = \arctan(\rho_1/\rho_2) , \quad (21)$$

which are respectively a measure the sizes of the  $\{12\}$  and  $\{T3\}$  clusters relative to the whole model universe and of the sizes of the  $\{12\}$  and  $\{T3\}$  clusters relative to each other. The corresponding RPM kinetic line element is

$$ds^2 = \{d\theta^2 + \sin^2\theta d\phi^2\}/2 . \quad (22)$$

(19), (22) and their  $\mathbb{S}^{n-1}$  generalization to ultraspherical coordinates is Key 3[ $N$ -stop].

Next, Key 5 [ $N$ -stop] is a useful set of redundant coordinates for a surrounding flat Euclidean space (here equal to relational space) are, for 4-stop metroland, simply the three relative Jacobi coordinate magnitudes,  $n^i$ .

### 5.3 Triangleland

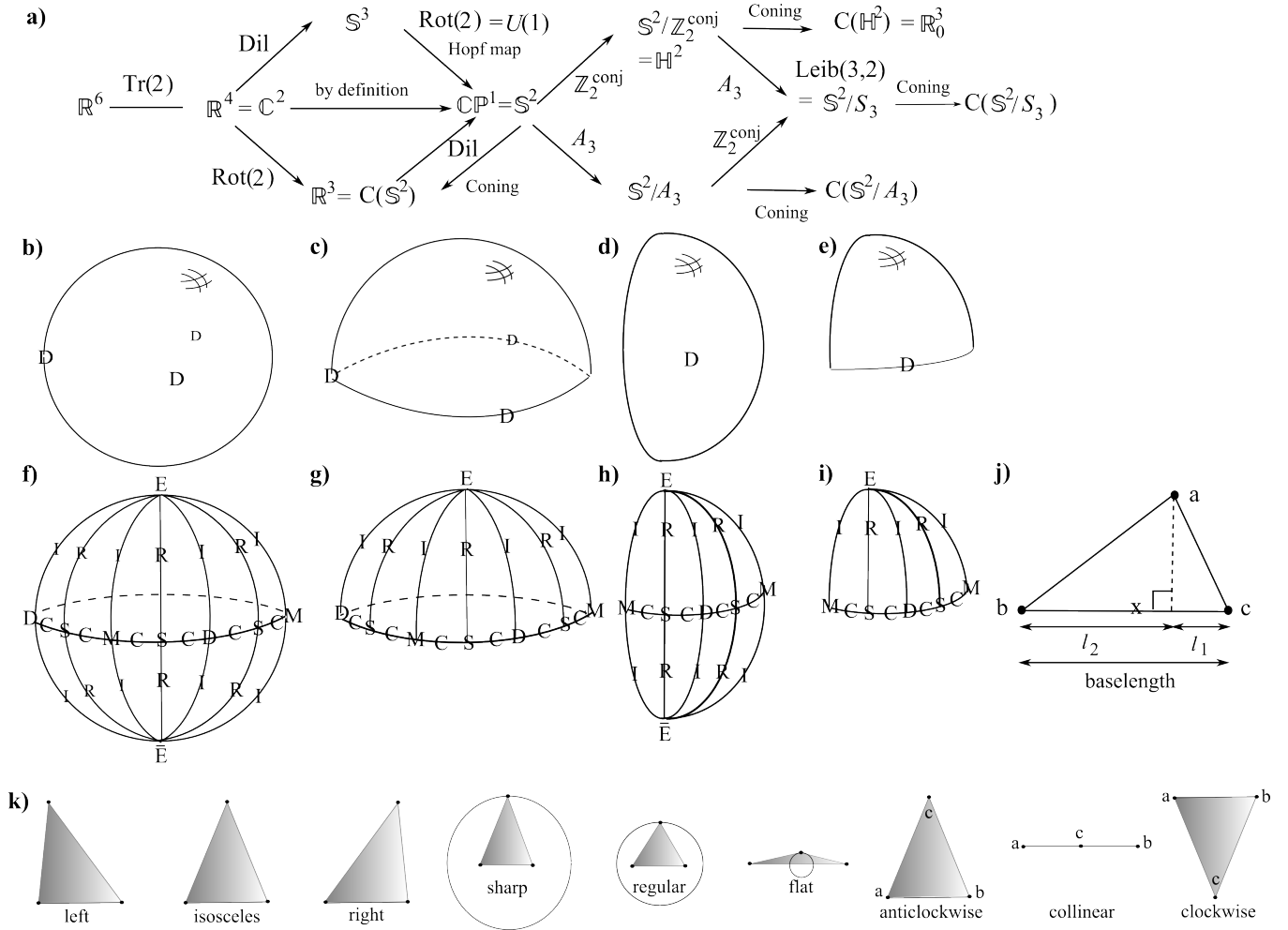


Figure 5: a) The sequence of configuration spaces for triangle-land.  $\mathbb{H}^2$  is the hemisphere with edge and  $\mathbb{R}_0^3$  the half-space with edge. b), c), d) and e) are the configuration spaces for, respectively, the distinguishable particle plain shape case, the distinguishable particle mirror-image-identified shape case, the indistinguishable particle plain shape case and the indistinguishable particle mirror-image-identified shape case at the topological level. f), g), h) and i) are the tessellation for the distinguishable particle plain shape case, the distinguishable particle mirror-image-identified shape case, the indistinguishable particle plain shape case and the indistinguishable particle mirror-image-identified shape case at the metric level. These pictures constitute [Key 7\[△\]](#). j) The anisoscelesness per unit base length is the amount by which the perpendicular to the base fails to bisect it is  $l_1 - l_2$  for  $l_1$  and  $l_2$  as indicated. k) The most physically meaningful great circles on the triangle-land shape correspond to the isosceles, regular and collinear triangles. These respectively divide the shape sphere into hemispheres of right and left triangles, sharp and flat triangles, and anticlockwise and clockwise triangles.

Here, we have  $\mathbb{S}^2$ , decorated as in Fig 5. Key 3[ $\triangle$ ] is that useful intrinsic diagonal coordinates for triangle-land are  $\Theta$  and  $\Phi$ . These are again spherical polar coordinates on the configuration space sphere, but their meaning in terms of the relative particle cluster separations is rather different. The interpretation of the azimuthal angle is now

$$\Theta = 2 \arctan(\rho_2/\rho_1) \, , \quad (23)$$

and that of the polar angle  $\Phi$  is as in Fig 1d). In terms of these, the triangleland kinetic line element is

$$ds^2 = \{d\Theta^2 + \sin^2\Theta d\Phi^2\}/2. \quad (24)$$

This is Key 3[ $\triangle$ ]. We also make use of the complex coordinate form of this which is a particular collapse of the kinetic line element built out of (16):

$$ds^2 = dZ d\bar{Z} / 2 \{1 + |Z|^2\}^2. \quad (25)$$

The configuration space here for distinguishable particles and in the plain shape case is the sphere, decorated as in Figs 5b) and 5f). The labelled points and edges have the following geometrical/mechanical interpretations.  $E$  and  $\bar{E}$  are the two mirror images of labelled equilateral triangles.  $C$  are arcs of the equator that is made up of collinear configurations. This splits the triangleland shape sphere into two hemispheres of opposite orientation (clockwise and anticlockwise labelled triangles, as in Fig 5c). Then it is clear that mirror-image-identification of shapes in space becomes, in the triangleland shape space, reflection in about the equator, a concept which immediately generalizes to  $N$  particles if reinterpreted as the complex conjugation operation. This produces the hemisphere with edge,  $\mathbb{H}^2$  [Fig 5c)], the cone over this then is  $\mathbb{R}_0^3$ :

the half-space with edge. The I are bimeridians of isoscelesness with respect to the 3 possible clusterings (i.e. choices of apex particle and base pair). Each of these separates the triangleland shape sphere into hemispheres of right and left slanting triangles with respect to that choice of clustering [Fig 5b)]. The R are bimeridians of regularness (equality of the 2 partial moments of inertia of the each of the possible 2 constituent subsystems: base pair and apex particle.) Each of these separated the triangleland shape sphere into hemispheres of sharp and flat triangles with respect to that choice of clustering [Fig 5a)]. The M are merger points: where one particle lies at the centre of mass of the other two. S denotes spurious points, which lie at the intersection of R and C but have no further notable properties (unlike the D, M or E points that lie on the other intersections).

Indistinguishability then involves quotienting out  $S_3 \cong \mathbb{D}_3$  (permutations of the particles, isomorphic to the dihaedral group acting on the triangle of double collision points D and distinguishing the 2 hemispheres) [Fig 5e) and 5i)]. If mirror-image identification is absent, rather one quotients out  $A_3 \cong \mathbb{Z}_3$  (even permutations of the particles, isomorphic to the cyclic group acting on the triangle of double collision points D) [Figs 5d) and 5i)].

Key 5[ $\triangle$ ] provides useful redundant coordinates covering a surrounding Euclidean space (here also the relational space) are now the complicated combinations of the two Jacobi vectors, namely Dragt coordinates

$$\text{dra}_1 = 2 \mathbf{n}_1 \cdot \mathbf{n}_2 \quad , \quad \text{dra}_2 = 2 \{ \mathbf{n}_1 \times \mathbf{n}_2 \}_3 \quad , \quad \text{dra}_3 = n_2^2 - n_1^2 \quad . \quad (26)$$

These are, respectively [46], the anisoscelesness aniso of the labelled triangle, four times the mass-weighted area of the triangle per unit moment of inertia and the ellipticity ellip of the two partial moments of inertia. The area Dragt coordinate is a democratic invariant and is useable as a measure of uniformity [72], its modulus running from maximal value at the most uniform configuration (the equilateral triangle) to minimal value for the collinear configurations. The on-sphere condition is then  $\sum_{A=1}^3 \text{dra}^A = 1$ . These combinations appearing as surrounding Cartesian coordinates is much less obvious than the  $n^e$  appearing in the same role for 4-stop metroland. These combinations arise from the sequence

$$\begin{array}{ccccccc} \mathbb{R}(3,2) & & \mathbb{P}(3,2) & & \mathbb{P}(3,2) & & \mathcal{R}(3,2) \\ = \mathbb{R}^4 & \xrightarrow{\text{obvious on-sphere condition}} & = \mathbb{S}^3 & \xrightarrow{\text{Hopf map}} & = \mathbb{S}^2 & \xrightarrow{\text{coning}} & = \mathbb{R}^3 \quad . \end{array} \quad (27)$$

Key 5b[ $\triangle$ ] One sometimes also swaps  $\text{dra}_2$  for the scale variable  $I$  in the non-normalized version of the coordinates to obtain the  $\{I, \text{aniso}, \text{ellip}\}$  system.

Interpretation of these coordinates is as follows.  $\text{dra}_3$  is an ‘ellipticity’, being the difference in the principal moments of inertia of the base and median. Also,  $\text{dra}_1$  can be interpreted as an ‘anisoscelesness’ (i.e. departure from isoscelesness, in analogy with anisotropy as a departure from isotropy in GR Cosmology); specifically, working in mass-weighted space, Aniso per unit base length is the amount by which the perpendicular to the base fails to bisect it ( $l_1 - l_2$  from Fig 3).  $\text{dra}_2$  has straightforward interpretation as  $4 \times \text{Area}$  (mass-weighted area per unit MOI); this is a measure of noncollinearity.

Key 6[ $\triangle$ ] Then a simple linear recombination of this is  $\{I_1, I_2, \text{aniso}\}$ , i.e. the two partial moments of inertia and the dot product of the two Jacobi vectors. This is in turn closely related [6] to the parabolic coordinates on the flat  $\mathbb{R}^3$  conformal to the triangleland relational space, which are  $\{I_1, I_2, \Phi\}$ .

Key 3b[ $\triangle$ ] Useful intrinsic coordinates are  $\Theta$  and  $\Phi$ . These are again spherical polars, but their meaning in terms of the relative particle cluster separations is rather different.

$$\Theta := 2 \arctan(\rho_2/\rho_1) \quad , \quad (28)$$

and  $\Phi$  is as in Fig 1’s caption.

## 5.4 Overview of configuration spaces for quadrilateralland

The shape space is  $\mathbb{CP}^2$  or some quotient as per Fig 6. N.B. this is harder to visualise than the previous subsections’ shape spaces, due to greater dimensionality as well as greater geometrical complexity and a larger hierarchy of special regions of the various possible codimensions. Some geometrical detail of this space is eliminated in Secs 9 and 10.

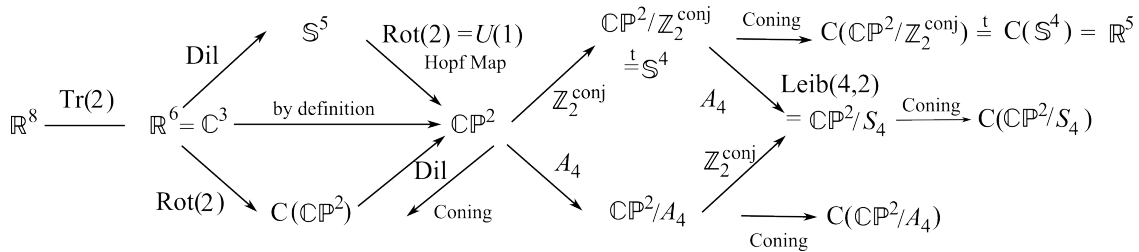


Figure 6: The sequence of configuration spaces for quadrilateralland.  $\stackrel{t}{=}$  denotes equality at the topological level.

## 6 Résumé of configuration spaces

This includes the GR-RPM analogies at the level of configuration spaces.

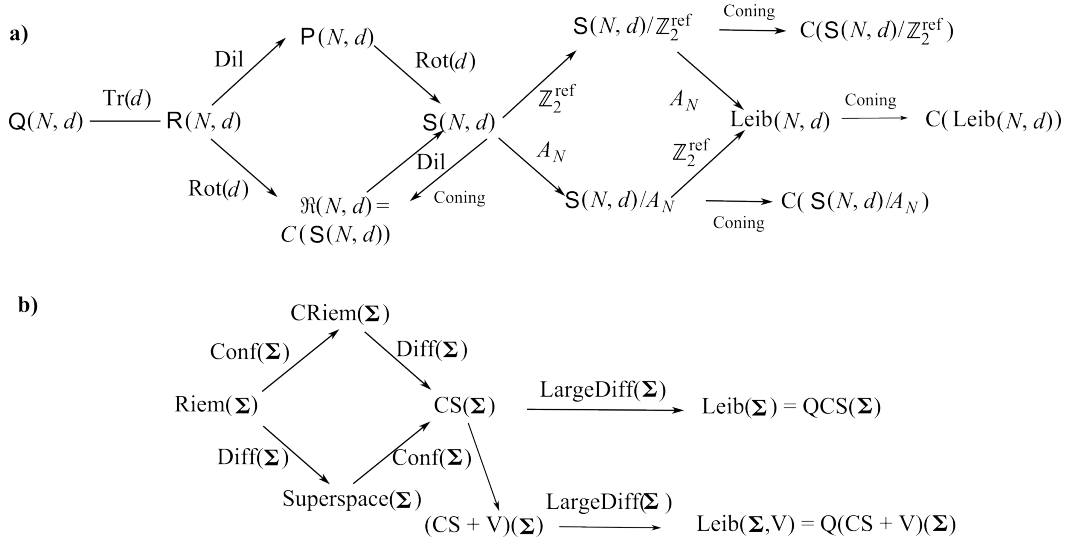


Figure 7: a) The sequence of configuration spaces for RPM's. b) The corresponding sequence of configuration spaces for GR.  $\text{Riem}(\Sigma)$  is the space of positive-definite 3 metrics on a fixed topology  $\Sigma$ , taken to be a compact without boundary one for simplicity. These most naturally correspond to relative space  $R(N, d)$ .  $\text{Diff}(\Sigma)$  and  $\text{Conf}(\Sigma)$  are the corresponding 3-diffeomorphisms and conformal transformations; these most naturally correspond to the rotations  $\text{Rot}(d)$  and dilations  $\text{Dil}$  respectively.  $\text{Superspace}(\Sigma)$  is meant in Wheeler's sense [75].  $\text{CRiem}(\Sigma)$  is pointwise superspace and  $\text{CS}(\Sigma)$  is conformal superspace.  $\text{Superspace}(\Sigma)$ ,  $\text{CRiem}(\Sigma)$  and  $\text{CS}(\Sigma)$  most naturally correspond to relational space  $\mathfrak{R}(N, d)$ , preshape space  $P(N, d)$  and shape space  $S(N, d)$  respectively. (RPM's also admit analogies with conformal/York-type [76] initial value problem formulations of GR, with the conformal 3-geometries playing here an analogous role to the pure shapes.)  $(\text{CS} + \text{V})(\Sigma)$  is conformal superspace to which has been adjoined a single global degree of freedom: the spatial volume of the universe.  $\text{CS}(\Sigma)$  and  $(\text{CS} + \text{V})(\Sigma)$  have on a number of occasions been claimed to be the space of true dynamical degrees of freedom of GR [76, 77]. The quotienting out of large diffeomorphisms gives the notion of quantum superspace and quantum conformal superspace,  $\text{QCS}(\Sigma)$ , as per [78]. This corresponds most naturally to identifying mirror image shapes and enforcing particle indistinguishability.  $\text{Leib}(N, d) = S(N, d)/S_N$  is then the analogue of  $\text{QCS}(\Sigma)$  and  $C(\text{Leib}(N, d))$  is then in some ways the analogue of quantum  $\text{CS} + \text{V}$ . It is named thus as the most very Leibnizian of the possible configuration spaces for mechanics with equal particle masses.

## 7 Collapses and ratio choices

These depend on the underlying tree (H or K) and furthermore on the choice of ratios. [Given a tree, which two ratios with a common denominator to make in passing to complex coordinates and then further coordinates whose interpretation depends on this prior choice.] The choice of ratios is furthermore related to collapses of the Jacobi vectors, which have geometrical significance as special, degenerate quadrilaterals; see Figure 8 for these degenerate quadrilaterals; a characterization I developed with Serna for the significance of the various choices of ratio is given below.

The basic H – denoted  $H(\text{DD})$  – involves ratios  $\rho_1/\rho_3$  and  $\rho_2/\rho_3$ .

The other type of ratio choice for the H – denoted  $H(\text{M}^*\text{D})$  – involves ratios  $\rho_2/\rho_1$  and  $\rho_3/\rho_1$ , or the  $1 \leftrightarrow 2$  permutation of this amounting to interchanging the two posts.

The basic K – denoted  $K(\text{T})$  – involves ratios  $\rho_1/\rho_2$  and  $\rho_3/\rho_2$ . This is convenient for addressing questions about the 3-particle subsystem picked out by the K-construction. This does then involve a less intuitive ‘post’ relative angle.

Another K – denoted  $K(\text{M}^*\text{D})$  – involves ratios  $\rho_3/\rho_1$  and  $\rho_2/\rho_1$ . This one succeeds in involving two knife relative angles at the expense of no longer focusing on the 3-particle subsystem.

The final type of ratio choice for the K – denoted  $K(\text{M}^{\text{T}}\text{D})$  – involves ratios  $\rho_1/\rho_3$  and  $\rho_2/\rho_3$ . This one is ‘as H-like as possible’, via using the seat much like the H uses the crossbar.

In terms of which Jacobi distances these involve (enumerated), the following transpositions hold.

$$H(\text{M}^*\text{D}) \text{ is the } 1 \leftrightarrow 2 \text{ of H and } K(\text{M}^*\text{D}) \text{ is the } 1 \leftrightarrow 3 \text{ of } K(\text{M}^{\text{T}}\text{D}) .$$

H and  $K(\text{M}^{\text{T}}\text{D})$  use the same ratios:  $K(\text{M}^{\text{T}}\text{D})$  is the closest realization of treating the chair as if it had 2 constituent subsystem posts (the back and leg) .

$$\text{Finally, } K(\text{T}) \text{ is the } 2 \leftrightarrow 3 \text{ of the preceding .} \quad (29)$$

These are useful in pinning various interpretations on coordinate systems and potentials.



submatrices have zero determinants, so that there are no more democracy invariants. For triangleland, however,  $\det(Q) = 4 \text{ Area}^2$  (for Area the area of the mass-weighted triangle per unit  $I$ ), while  $Q_{ij}$  is but  $2 \times 2$ , so that there are no more invariants. For quadrilateralland,  $\det(Q) = 0$ . This can be understood in terms of volume forms being zero for planar figures. Also, being  $3 \times 3$ , there is one further invariant,  $Q_{11}Q_{22} - Q_{12}^2 + \text{cycles} = \rho_1^2 \rho_2^2 - \{\rho_1 \cdot \rho_2\}^2 + \text{cycles} = |\rho_1 \times \rho_2|_3^2 + \text{cycles}$ , i.e. proportional to the sum of squares of areas of various coarse-graining triangles (see Fig 2). Here the 3-suffix denotes that it is mathematically the component in a fictitious third dimension of the cross product. E.g. in the  $\{12, 34\}$  H-clustering, these are the coarse-grainings by taking 1, 2 and COM(34); 3, 4 and COM(12); 12 and 34 with one shifted so that its COM coincides with the other's.

## 8.1 Shape quantities and democracy invariants for triangleland

I present here a different method of finding the useful Dragt-type shape coordinates for triangleland, a method which I find to extend to the quadrilateralland case, as well as providing a useful prequel for this as regards notation and interpretation.  $\underline{\underline{Q}}$  may be written as  $\frac{1}{2}\{w_1 \underline{\underline{1}} + w_2 \underline{\underline{2}} + w_3 \underline{\underline{3}}\}$  (for Pauli matrices  $\underline{\underline{2}}$  and  $\underline{\underline{3}}$ ;  $\underline{\underline{1}}$  does not feature since  $\underline{\underline{Q}}$  is symmetric). This takes such an  $SU(2)$  form due to the 3-particle case being exceptional through the various tensor fields of interest possessing higher symmetry in this case [52]. Then, reading off from the definition of  $\underline{\underline{Q}}$  [and dropping (1)-labels],

$$w = \rho^2 = I(\text{moment of inertia}) := \text{Size} , \quad (32)$$

$$w_1 = \rho_1^2 - \rho_2^2 := \text{Ellip} , \quad (33)$$

$$w_3 = 2 \rho_1 \cdot \rho_2 := \text{Aniso} . \quad (34)$$

There is also a

$$w_2 = 2 |\rho_1 \times \rho_2|_3 \quad (35)$$

such that  $w^2 = \sum_{A=1}^3 w^A{}^2$ . (35) has straightforward interpretation as  $4 \times \text{Area}$ .

For triangleland,  $w_1$ ,  $w_2$  and  $w_3$  are, again, Dragt-type coordinates. For these to be shape quantities, I divide each of these by  $I$  and switch the 2 and 3 labels (so the principal axis is aligned with the democracy invariant), obtaining

$$s_1 = n_1^2 - n_2^2 = \text{Ellip}/I := \text{ellip} , \quad (36)$$

$$s_2 = 2 \mathbf{n}_1 \cdot \mathbf{n}_2 = \text{Aniso}/I := \text{aniso} , \quad (37)$$

$$s_3 = 2 |\mathbf{n}_1 \times \mathbf{n}_2|_3 = 4 \times \text{Area}/I := 4 \times \text{area} := \text{demo}(N=3) . \quad (38)$$

(area is clearly going to be clustering invariant and thus a democratic invariant). These are in  $\mathbb{R}^3 = C(\mathbb{S}^2)$  and subject to the on- $\mathbb{S}^2$  restriction  $\sum_{A=1}^3 s^A{}^2 = 1$ . Interpretation of various great circles and of the hemispheres they divide triangleland into are provided in Fig 5.

## 8.2 Shape quantities and democratic invariants for quadrilateralland

Repeating this for quadrilateralland closely parallels the analysis for the tetrahaedron in [52] because the democracy group does not care about dimension. Now,

$$\underline{\underline{Q}} = \left\{ w \underline{\underline{1}} + \sum_{\delta=1}^5 w_{\delta} \underline{\underline{B}}_{\delta} \right\} / 2 , \quad (39)$$

where the  $\underline{\underline{B}}_{\delta}$  are  $j=1$  representation matrices of the  $SO(3)$  democracy group (they are also proportional to the real symmetric subset of the Gell-Mann  $\lambda$ -matrices),

$$\underline{\underline{B}}_1 = \sqrt{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \underline{\underline{B}}_2 = \sqrt{3} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \underline{\underline{B}}_3 = \sqrt{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \underline{\underline{B}}_4 = \sqrt{3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \underline{\underline{B}}_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} . \quad (40)$$

Thus, reading off from the definition of  $\underline{\underline{Q}}$  and dropping (H2) labels,

$$w = Q_{11} + Q_{22} + Q_{33} = \rho_1^2 + \rho_2^2 + \rho_3^2 = \rho^2 = I : \text{moment of inertia (MOI)} , \quad (41)$$

$$w_1 = \sqrt{3} \{Q_{11} - Q_{22}\} / 2 = \sqrt{3} \{\rho_1^2 - \rho_2^2\} / 2 = \sqrt{3} \text{Ellip}(12) / 2 , \quad (42)$$

$$w_2 = \sqrt{3} Q_{12} = \sqrt{3} \rho_1 \cdot \rho_2 = \sqrt{3} \text{Aniso}(12) / 2 , \quad (43)$$

$$w_3 = \sqrt{3} Q_{23} = \sqrt{3} \rho_2 \cdot \rho_3 = \sqrt{3} \text{Aniso}(23) / 2 , \quad (44)$$

$$w_4 = \sqrt{3} Q_{31} = \sqrt{3} \rho_3 \cdot \rho_1 = \sqrt{3} \text{Aniso}(31) / 2 , \quad (45)$$

$$w_5 = \{-Q_{11} - Q_{22} + 2Q_{33}\} / 2 = \{-\rho_1^2 - \rho_2^2 + 2\rho_3^2\} / 2 = \{\text{Ellip}(31) + \text{Ellip}(32)\} / 2 . \quad (46)$$

Here, the brackets refer to the pair of Jacobi vectors involved in each coarse-graining triangle (Fig 2); note that Ellip being a signed quantity requires this to be an ordered pair.

Note also that if one sets

$$w_6 = \sqrt{3}\{|\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2|_3^2 + \text{cycles}\}^{1/2} = \sqrt{12}\{\text{Area}(12)^2 + \text{cycles}\}^{1/2}, \quad (47)$$

i.e. a quantity proportional to the square root of the above-found democracy invariant, then  $w^2 = \sum_{\mathbf{A}=1}^6 w^{\mathbf{A}^2}$ . One can then straightforwardly make the  $w^{\mathbf{A}}$  into surrounding shape quantities by dividing through by  $I$ :

$$s_1 := \sqrt{3}\{n_1^2 - n_2^2\}/2 = \sqrt{3}\text{ellip}(12)/2 \quad (48)$$

$$s_2 := \sqrt{3}\mathbf{n}_1 \cdot \mathbf{n}_2 = \sqrt{3}\text{aniso}(12)/2, \quad (49)$$

$$s_3 := \sqrt{3}\mathbf{n}_2 \cdot \mathbf{n}_3 = \sqrt{3}\text{aniso}(23)/2, \quad (50)$$

$$s_4 := \sqrt{3}\mathbf{n}_3 \cdot \mathbf{n}_1 = \sqrt{3}\text{aniso}(31)/2, \quad (51)$$

$$s_5 := \{-n_1^2 - n_2^2 + 2n_3^2\}/2 = \{\text{ellip}(31) + \text{ellip}(32)\}/2, \quad (52)$$

$$s_6 := \sqrt{3}\{|\mathbf{n}_1 \times \mathbf{n}_2|_3^2 + \text{cycles}\}^{1/2} = \sqrt{12}\{\text{area}(12)^2 + \text{cycles}\}^{1/2} = \text{demo}(N=4). \quad (53)$$

These belong to  $\mathbb{R}^6 = \mathbb{C}^3$  and the relation between the  $w^{\mathbf{A}}$  becomes the on- $\mathbb{S}^5$  condition  $\sum_{\mathbf{A}=1}^6 s^{\mathbf{A}^2} = 1$ . That an  $\mathbb{S}^5$  makes an appearance is not unexpected, from Fig 6 and using the generalization of the Hopf map in the sense of  $U(1) = SO(2)$  fibration,

$$\begin{array}{ccccc} \mathbb{R}(4,2) & & \mathbb{P}(3,2) & & \mathbb{P}(3,2) & & \mathcal{R}(4,2) \\ = \mathbb{R}^6 & \xrightarrow{\text{obvious on-sphere condition}} & = \mathbb{S}^5 & \xrightarrow{\text{Hopf map}} & = \mathbb{CP}^2 & \xrightarrow{\text{coning}} & = C(\mathbb{CP}^2) \end{array} \quad (54)$$

Moreover, since  $\mathbb{CP}^2$  is 4- $d$  and  $\mathbb{S}^5$  is 5- $d$ , there is a further condition on the  $s^{\mathbf{A}}$  for quadrilateralland. Also, the  $C(\mathbb{CP}^2)$  is 5- $d$  but the shape coordinates are 6- $d$ . Also above,  $\text{aniso}(12)$  is same notion of anisoscelesness as before but now applied to the triangle made out of the 1 and 2 relative Jacobi vectors, and so on. Finally, note the interpretation

$$\text{demo}(N=4) = \sqrt{12} \times \text{sum of squares of mass-weighted areas of coarse-graining triangles per unit MOI}. \quad (55)$$

Quadrilateralland's relational space is locally  $\mathbb{R}^5$ . One can envisage this from  $\mathbb{CP}^2/\mathbb{Z}_2 = \mathbb{S}^4$  [79] and then taking the cone. As we are treating  $\mathbb{CP}^2$ , we have, rather, two copies of  $\mathbb{S}^4$ . Such a doubling does not appear in the coplanar boundary of [52]'s study of 4 particles in 3- $d$ , since 3- $d$  dictates that the space of shapes on its coplanar boundary is the mirror-image-identified one. Another distinctive feature of 3- $d$  is that it has a third volume-type democracy invariant,  $\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3$ .

Finally,  $\text{demo}(N=4) = s_6$  can be used to diagnose collinearity as the points at which this is zero. It is in this respect like triangleland's  $\text{demo}(N=3) = \text{dra}_2$ , which also gives the equilateral configurations (corresponding to a manifest notion of maximum uniformity) as the output to its extremization, corresponding to the natural  $\mathbb{S}^2$  poles. However, one can quickly see that extremizing  $\text{demo}(N=4)$  will involve something more complicated, since there are not two but six permutations of labelled squares (likewise a manifest notion of maximum uniformity), so these cannot possibly all lie on the two  $\mathbb{S}^5$  poles, nor could any of them lie on the poles by a symmetry argument. In fact, extremizing  $\text{demo}(N=4)$  does yield these squares, but not uniquely. One gets, rather, one curve of extrema for each of the two orientations, each containing three labelled squares.

Key 5[□]  $\{s^{\mathbf{A}}\}$  are a redundant set of six shape coordinates (see [72] for their explicit forms), which are the quadrilateralland analogue of triangleland's Dragt coordinates according to the construction in [52, 72]. These include one democracy invariant as per above.

## 9 Further Useful coordinate systems for quadrilateralland.

### 9.1 The $\{I, s^{\mathbf{A}}\}$ coordinate system

However, for quadrilateralland, swapping one of the  $s^{\mathbf{A}}$  [ $s_6 = \text{demo}(4)$ ], a democracy invariant and proportional to the square root of the sum of the squares of the mass-weighted areas per unit MOI of the three coarse-graining triangles and parallelogram of Fig 8b-d) and g-i)] for a scale variable to form the coordinate system  $\{I, s^{\mathbf{A}}\}$  turns out to give a more useful set [72].

### 9.2 Key 6[□] Kuiper coordinates

These are then a simple linear combination of 2) (mixing the  $I$ ,  $s_1$  and  $s_5$  coordinates). These consist of all the possible inner products between pairs of Jacobi vectors, i.e. 3 magnitudes of Jacobi vectors per unit MOI  $N^i$ , alongside 3  $\mathbf{n}^i \cdot \mathbf{n}^j$  ( $i \neq j$ ), which are very closely related to the three relative angles. As such, they are, firstly, very much an extension of the parabolic coordinates for the conformally-related flat  $\mathbb{R}^3$  of triangleland [6]. Secondly, in the quadrilateralland setting, they are a clean split into 3 pure relative angles (of which any 2 are independent and interpretable as the anisoscelesnesses of the coarse-graining triangles and one coarse-graining rhombus) and 3 magnitudes (supporting 2 independent non-angular ratios). I therefore denote this coordinate system by  $\{N^i, \text{aniso}(i)\}$ . Thirdly, whilst they clearly contain 2 redundancies, they are fully democratic in relation to the constituent Jacobi vectors and coarse-graining triangles/parallelogram made from pairs of them.



### 9.3 Key 3b[□]: Gibbons–Pope coordinates

The Gibbons–Pope type coordinates [69]  $\{\chi, \beta, \phi, \psi\}$  are useful intrinsic coordinates for quadrilateralland in a number of senses that extend the rule of the spherical coordinates on the triangleland and 4-stop metroland shape spheres). These are a subcase of Euler angle adapted coordinates for a privileged  $SU(2)$  subgroup of the  $SU(3)$  is picked out, also see [80] and adapted coordinate systems along these lines were probably known previously to that. In fact they pick out one of the three constituent overlapping  $SU(2) \times U(1)$ 's worth of conserved quantities in simple and clear form. These coordinates additionally place the metric in as diagonal a form as is possible (Fig 9).

Figure 9: The block structure type of the Gibbons–Pope type coordinates.

This makes the form of the remaining coordinate less important (i.e. is there is a wider class of  $SU(2)$ -Euler-adapted coordinate systems with this same type of block structure for the different choices of the last coordinate). The particular Gibbons–Pope type choice of the  $\chi$  further tidies the constituent blocks.

The coordinate ranges are  $0 \leq \chi \leq \pi/2$ ,  $0 \leq \beta \leq \pi$ ,  $0 \leq \phi \leq 2\pi$  (a reasonable range redefinition since it is the third relative angle), and  $0 \leq \psi \leq 4\pi$ . These are related to the bipolar form of the Fubini–Study coordinates by

$$\psi' = -\{\Phi_1 + \Phi_2\}, \quad \phi' = \Phi_2 - \Phi_1, \quad (56)$$

with then  $\psi = -\psi'$  (measured in the opposite direction to match Gibbons–Pope's convention) and  $\phi$  is taken to cover the coordinate range 0 to  $2\pi$ , which is comeasurate with it itself being the third relative angle between the Jacobi vectors involved.

$$\beta = 2 \arctan(\mathcal{R}_2/\mathcal{R}_1), \quad \chi = \arctan(\sqrt{\mathcal{R}_1^2 + \mathcal{R}_2^2}). \quad (57)$$

From here on, a different interpretation is to be attached to these last two formulae in terms of the  $\rho_i$ , according to the choice of H or K and of ratios.

**General interpretation.** By their ranges,  $\beta$  and  $\phi$  parallel azimuthal and polar coordinates on the sphere. [In fact,  $\beta$ ,  $\phi$  and  $\psi$  take the form of Euler angles on  $SU(2)$ , with the remaining coordinate  $\chi$  playing the role of a compactified radius.] Now, in the quadrilateralland interpretation,  $\beta$  has the same mathematical form as triangleland's azimuthal coordinate  $\Theta$  (23). Additionally,  $\chi$  parallels 4-stop metroland's azimuthal coordinate  $\theta$  [the first equation in (20)], except that it is over half of the range of that, since the collinear 1234 and 4321 orientations have to be the same due to the existence of the second dimension via which they can be rotated into each other.

**Specific interpretation by tree and ratio choice.** The basic H = H(DD)'s  $\beta$  is then contents inhomogeneity variable for the ratio of the two posts, whilst its  $\chi$  is what proportion of the universe is occupied by its post contents. Its  $\phi$  and  $\psi$  are a sum and a difference of knife angles. These may be less intuitive than the big  $\Phi$ 's, but their conjugate momenta are revealed to have a more lucid interpretation in Paper II.

$$\beta = 2 \arctan(\rho_2/\rho_1), \quad \chi = \arctan(\sqrt{\rho_1^2 + \rho_2^2}/\rho_3). \quad (58)$$

H(M\*D)'s  $\beta$  is then the ratio of the selected post to the crossbar, whilst its  $\chi$  is the ratio of the selected-post-and-crossbar to the non-selected post. Its  $\phi$  and  $\psi$  are a sum and a difference of a knife angle and a post angle.

The basic K = K(T)'s  $\beta$  is then the ratio of the back to the seat (i.e. a sharpness/flatness shape quantity for the obvious triangle subsystem), whilst  $\chi$  is the ratio of the back-and-seat (which in  $d > 1$  forms the obvious triangle subsystem) to the leg. In this case,

$$\beta = 2 \arctan(\rho_1/\rho_3), \quad \chi = \arctan(\sqrt{\rho_1^2 + \rho_3^2}/\rho_2). \quad (59)$$

Its  $\phi$  and  $\psi$  are a sum and a difference of knife angles.

K(M\*D)'s  $\beta$  is then the ratio of the seat to the leg, whilst its  $\chi$  is the ratio of the seat-and-leg to the back. Its  $\phi$  and  $\psi$  are a sum and a difference of a knife angle and a post angle.

K(M<sup>T</sup>D)'s  $\beta$  is then the ratio of the back to the leg, whilst its  $\chi$  is the ratio of the back-and-leg to the seat. Its  $\phi$  and  $\psi$  are a sum and a difference of a knife angle and a post angle.

Finally, in (whichever, with labels suppressed) Gibbons–Pope type coordinates, the Fubini–Study line element takes the form

$$ds^2 = d\chi^2 + \sin^2\chi \{d\beta^2 + \cos^2\chi \{d\phi^2 + d\psi^2 + 2 \cos\beta d\phi d\psi\} + \sin^2\chi \sin^2\beta d\phi^2\} / 4. \quad (60)$$

As promised, this is of the form of Fig 9.

## 10 Inclusion of trianglelands, 4-stop metroland within quadrilateralland

This concerns Key 7 within restrictions due to the shape space now being 4- $d$  and thus harder to visualize.

In Gibbons–Pope type coordinates based on the Jacobi H, when  $\rho_3 = 0$ ,  $\chi = \pi/2$  and the metric reduces to

$$ds^2 = \{1/2\}^2 \{d\beta^2 + \sin^2 \beta d\phi^2\} \quad (61)$$

i.e. a sphere of radius  $1/2$ , which corresponds to the conformally-untransformed  $\mathbb{CP}^1$ . When  $\rho_2 = 0$ ,  $\beta = 0$  and the metric reduces to

$$ds^2 = \{1/2\}^2 \{d\Theta_1^2 + \sin^2 \Theta_1 d\Phi_1^2\} \quad (62)$$

for  $\Theta_1 = 2\chi$  having the correct coordinate range for an azimuthal angle. Finally, when  $\rho_1 = 0$ ,  $\beta = 0$  and the metric reduces to

$$ds^2 = \{1/2\}^2 \{d\Theta_2^2 + \sin^2 \Theta_2 d\Phi_2^2\} \quad (63)$$

for  $\Theta_2 = 2\chi$  again having the correct coordinate range for an azimuthal angle. The first two of these spheres are a triangleland shape sphere included within quadrilateralland. The first and second of these are, respectively, collapses to the spaces of the triangles of Fig 8e) and 8f), whilst the third of these is collapse to the space of rhombi of Fig 8g).

In Gibbons–Pope type coordinates based on the Jacobi K, when  $\rho_3 = 0$ ,  $\chi = \pi/2$  and the metric reduces to

$$ds^2 = \{1/2\}^2 \{d\beta^2 + \sin^2 \beta d\phi^2\} \quad (64)$$

i.e. a sphere of radius  $1/2$ , which corresponds to the conformally-untransformed  $\mathbb{CP}^1$ . When  $\rho_2 = 0$ ,  $\beta = 0$  and the metric reduces to

$$ds^2 = \{1/2\}^2 \{d\Theta_1^2 + \sin^2 \Theta_1 d\Phi_1^2\} \quad (65)$$

for  $\Theta_1 = 2\chi$  again having the correct coordinate range for an azimuthal angle. Finally, when  $\rho_1 = 0$ ,  $\beta = 0$  and the metric reduces to

$$ds^2 = \{1/2\}^2 \{d\Theta_2^2 + \sin^2 \Theta_2 d\Phi_2^2\} \quad (66)$$

for  $\Theta_2 = 2\chi$  yet again having the correct coordinate range for an azimuthal angle. The first, second and third of these are, respectively, the collapses to the spaces of the triangles of Fig 8l), 8j) and 8k).

In Kuiper coordinates, each of the three on- $\mathbb{S}^2$  conditions, for whichever of H or K coordinates, involves losing one magnitude and two inner products. Thus the survivors are two magnitudes and one inner product (closely related to parabolic coordinates and linearly combineable to form the  $\{I, \text{aniso}, \text{ellip}\}$  system, see Sec 4). If we recombine the Kuiper coordinates to form the  $\{I, s^a\}$  system, and swap the  $I$  for the  $s_6 = \text{demo}(4)$ , then the survivors of the procedure are the Dragt coordinates (since the procedure kills two of the three area contributions to the  $s_6$ ). This gives another sense in which the  $\{s^A\}$  system is a natural extension of the Dragt system.

Quadrilateralland is also decorated by a net of 6  $\mathbb{S}^2$  trianglelands (in each case with one vertex being a double collision).

While it is also easy to write conditions in these coordinates for rectangles, kites, trapezia, rhombi... , I comment that these are less meaningful 1) from a mathematical perspective (e.g. they are not topologically defined and are less related to the underlying constellation) and 2) as regards their physical meaningfulness.

### 10.1 The split into hemi- $\mathbb{CP}^2$ 's of oriented quadrilaterals

**Definition:** The Veronese surface  $V$  [81] is the space of conics through a point (parallel to how a projective space is a set of lines through a point).

**Kuiper's Theorem i)** The map

$$\eta : (z_1, z_2, z_3) \in \mathbb{CP}^2 \longmapsto (|z_1|^2, |z_2|^2, |z_3|^2, \{z_2 \bar{z}_3 + z_3 \bar{z}_2\}/2, \{z_3 \bar{z}_1 + z_1 \bar{z}_3\}/2, \{z_1 \bar{z}_2 + z_2 \bar{z}_1\}/2) \in \mathbb{E}^5 \quad (67)$$

induces a piecewise smooth embedding of  $\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}$  onto the boundary of the convex hull of the Veronese surface  $V$  in  $\mathbb{E}^5$ , which moreover has the right properties to be the usual smooth 4-sphere [79].

N.B. this is at the topological level; it clearly cannot extend to the metric level by a mismatch in numbers of Killing vectors ( $\mathbb{S}^4$  with the standard spherical metric has 10 whilst  $\mathbb{CP}^2$  equipped with the Fubini–Study metric has 8).

Restricting  $\{z_1, z_2, z_3\}$  to the real line corresponds in the quadrilateralland interpretation to considering the collinear configurations, which constitute a  $\mathbb{RP}^2$  space as per Sec 5.2 Moreover the above embedding sends this onto the Veronese surface  $V$  itself. Proving this proceeds via establishing that, as well as the on- $\mathbb{S}^5$  condition

$$\sum_{i=1}^3 N^i = \sum_{i=1}^3 n^{i2} = 1, \quad (68)$$

a second restriction holds, which in our quadrilateralland interpretation, reads

$$\sum_{i=1}^3 \text{aniso}(i)^2 N^i - 4N_1 N_2 N_3 + \text{aniso}(1) \text{aniso}(2) \text{aniso}(3) = 0 \quad (69)$$

(Knowledge of this restriction is also useful in kinematical quantization [82], and it is clearer in the  $\{N^i, \text{aniso}(i)\}$  system, which is both the quadrilateralland interpretation of Kuiper's redundant coordinates and a simple linear recombination of the shape coordinates  $\{I, s^a\}$  coordinates.)

**Another form for Kuiper's theorem** [79]. Moreover,  $\mathbb{CP}^2$  itself is topologically a double covering of  $\mathbb{S}^4$  branched along the  $\mathbb{RP}^2$  of collinearities which itself embeds onto  $\mathbb{E}^5$  to give the Veronese surface  $V$ .

Here, branching is meant in the sense familiar from the theory of Riemann surfaces [84]. Moreover, the  $\mathbb{RP}^2$  itself embeds *non-smoothly* into the Veronese surface  $V$ .

Then the quadrilateralland interpretation of these results is in direct analogy with the plain shapes case of triangle and consisting of two hemispheres of opposite orientation bounded by an equator circle of collinearity, the mirror-image-identified case then consisting of one half plus this collinear edge. Thus plain quadrilaterallands distinction between clockwise- and anticlockwise-oriented figures is strongly anchored to this geometrical split, with the collinear configurations lying at the boundary of this split.

## 11 Kuiper coordinates in terms of Gibbons–Pope coordinates

This is motivated by the Kuiper quantities occurring in the HO potentials (see Paper II) as well as furnishing interesting ‘shape operators’ at the quantum level. In each case, however, the kinetic part of the Hamiltonian is far more conveniently expressed in terms of Gibbons–Pope-type coordinates, thus necessitating the below conversions:

$$N_3 = \cos^2 \chi, \quad (70)$$

$$N_1 = \sin^2 \chi \cos^2 \frac{\beta}{2}, \quad (71)$$

$$N_2 = \sin^2 \chi \sin^2 \frac{\beta}{2}, \quad (72)$$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \sin^2 \chi \sin \beta \cos \phi =: \sin^2 \chi \sin \beta \cos f_3, \quad (73)$$

$$\mathbf{n}_2 \cdot \mathbf{n}_3 = \sin 2\chi \sin \frac{\beta}{2} \cos \frac{\psi+\phi}{2} =: \sin 2\chi \sin \frac{\beta}{2} \cos f_1, \quad (74)$$

$$\mathbf{n}_3 \cdot \mathbf{n}_1 = \sin 2\chi \cos \frac{\beta}{2} \cos \frac{\psi-\phi}{2} =: \sin 2\chi \sin \frac{\beta}{2} \cos f_2. \quad (75)$$

[The  $f_i$  form is useful as regards transposition symmetry arguments interrelating interpretations in terms of different ratio choices.] I also note that these are a nice rendition of quantities that auto-obey the two on- $\mathbb{CP}^2$  conditions as trigonometric *identities*, which plays a role in Paper III's quantum scheme. (68) obviously holds, whilst (69) becomes

$$\sum_{i=1}^3 \cos^2 f_i + 2 \cos f_1 \cos f_2 \cos f_3 = 1, \quad (76)$$

which trigonometric identity indeed holds for angles such that  $f_1 + f_2 + f_3 = 0$ , which is true since we are free to change signs of some of the  $f_i$  here since they occur solely inside cos functions for which signs do not matter. Going from (69) to (76) amounts to reversing the working by which Kuiper obtained the non-quadrilateralland-interpreted version of (69) in the first place.

What of the shape quantities?  $s_2, s_3$  and  $s_4$  are just  $\sqrt{3} \times (70-72)$  respectively.

$$s_1 = \sqrt{3} \sin^2 \chi \cos \beta / 2, \quad (77)$$

$$s_5 = \{3 \cos^2 \chi - 1\} / 2 \quad \text{and} \quad (78)$$

$$\text{demo}(N=4)^2 = s_6^2 = 3 \sin^2 \chi \{ \sin^2 \chi \sin^2 \beta \sin^2 \phi / 4 + \cos^2 \chi \{ \sin^2 \frac{\beta}{2} \sin^2 \frac{\psi+\phi}{2} + \cos^2 \frac{\beta}{2} \sin^2 \frac{\psi-\phi}{2} \} \}. \quad (79)$$

Note 1)  $s_5$  is the second Legendre polynomial in the azimuthal angle analogue  $\chi$ .

Note 2) The Kuiper coordinates have the advantage of having a triple of  $\phi$  and  $\psi$  independent quantities, whilst the  $s^A$  trade one of these for the far more complicated  $s_6$ . Though it should be said that the  $s_6$  is an interesting quantity in its own right; I suggest the squared version of this be used since it is more readily insertible into quantum integrals though being just a sum of products of Kuiper coordinates,

$$\text{demo}(N=4)^2 = 3\{N_1 N_2 + N_3\{1 - N_3\} - \sum_{i=1}^3 \text{aniso}(i)/16 \quad (80)$$

(rather than containing an overall square root).

Note 3) There is also relevance to quantities such as the individual areas, e.g.

$$\sin^2 \chi \sin \beta \sin \phi / 2, \quad (81)$$

for one of the coarse-graining triangle subsystems. These are mathematically simpler than the whole democratic invariant, albeit this is at the cost of them being in the context of particular subsystems associated with particular clusterings rather than whole-universe properties that are clustering-independent.

## 12 Notions of uniformity for quadrilateralland

In this quadrilateralland case, there are then 3 squares in each hemi- $\mathbb{CP}^2$  as opposed to the single equilateral triangle in each hemisphere of triangleland; these are particularly uniform configurations. This reflects the presence of a further 3-fold symmetry in quadrilateralland; choosing to use indistinguishable particles then quotients this out.

As explained in detail in [46], the nontrivial democratic invariant  $\text{demo}(3)$  for triangleland is the mass-weighted area of the triangle per unit moment of inertia, and that for quadrilateralland,  $\text{demo}(4)$ , is proportional to the square root of the sum of the mass-weighted areas of the coarse-graining triangles/rhombi in Fig 8. Now, extremizing  $\text{demo}(3)$  invariant picks out the two labelled equilateral triangles, which is uniform in a very strong sense; on the other hand, extremizing  $\text{demo}(4)$  does pick out the six labelled squares, but nonuniquely – one gets one extremal curve per hemi- $\mathbb{CP}^2$ . Now, the present paper’s choices of coordinate systems furthermore clarify the interpretation to be given to these two extremal curves. In  $\{N^i, \text{aniso}(i)\}$  coordinates, this is given by  $\text{aniso}(1) = 0 = \text{aniso}(2)$  and  $|\text{aniso}(3)| = 1$ , i.e. (1)-isosceles, (2)-isosceles and maximally (3)-right or (3)-left i.e. (3)-collinear, with  $N_3 = 1/2$  and  $N_1, N_2$  varying (but such that the on- $\mathbb{S}^5$  condition  $N_1 + N_2 + N_3 = 1$  holds). On the other hand, in Gibbons–Pope type coordinates, the uniformity condition is  $\phi = 0, \psi = 4\pi, \chi = \pi/4$  and  $\beta$  free, i.e., in the H-coordinates case, freedom in the contents inhomogeneity i.e. size of subsystem 1 relative to the size of subsystem 2. On the other hand, in the K-coordinates case,  $\beta$ -free signifies freedom in how tall one makes the selected  $\{12, 3\}$  cluster’s triangle.

While  $\text{demo}(N = 4)$  for quadrilateralland is as good a diagnostic as  $\text{demo}(N = 3)$  for triangleland as regards collinear configurations, that the extrema of  $\text{demo}(N = 3)$  pick out the triangleland uniform states (the 2 labellings of equilateral triangle) only partly carries over to the extrema of  $\text{demo}(N = 4)$ , since these do not uniquely pick out the quadrilateral’s most uniform states (the 6 labellings of the square).

## 13 Conclusion

### 13.1 Outline of the results so far

Quadrilateralland is the smallest model that possesses both linear constraints (conferring it a ‘midisuperspace toy model character’ in the cosmologically-significant case with scale) and nontrivial subsystem structure. The latter includes two non-overlapping subsystems of two particles ( $2 + 2$  split) and subsystems of 2 particles nested within subsystems of 3 particles ( $\{2 + 1\} + 1$  split). Applications of this include investigation of Records Theory (see Sec 13.3), structure formation (see Paper IV) and investigating the extent to which one subsystem can be used as a clock for the other parts of the model universe [38, 5]. These are useful in the timeless records, histories, semiclassical and combined Halliwell-type approaches to the Problem of Time considered in Paper IV (and make for interesting questions in the Naïve Schrödinger Interpretation approach posed below and resolved once one has the form of the wavefunctions in Paper III).

Counterparts of the configuration space level Keys that solved triangleland [20, 48, 12, 6, 46] are as follows for quadrilateralland. (These answer the questions at the end of Sec 1.3).

Key 1: the Jacobi coordinates are available to diagonalize relative interparticle cluster coordinates regardless of particle number  $N$  or dimension. For quadrilaterals, these are two qualitatively different choices of Jacobi tree of clusterings: H-shaped and K-shaped. These are, respectively, adapted to  $2 + 2$  and  $\{2 + 1\} + 1$  splits of the particles into subsystem clusters.

Key 2: Any  $N$ -a-gonland’s shape space is  $\mathbb{CP}^{N-2}$  with the Fubini–Study metric on it (for the quadrilateralland case it is  $\mathbb{CP}^2$ ), or some or some quotient of this space by a discrete group involving reflection and/or permutations of the particles so as to model indistinguishability. In the quadrilateralland case in which distinct particle masses do not naturally label the particles as distinct, the most Leibnizian shape space,  $\text{Leib}(4, 2)$ , is  $\mathbb{CP}^2/S_4$ .

Key 3 is, in part, the expression of the previous in inhomogeneous coordinates.

The remaining part of Key 3 – Key 3b – is the expression of the Fubini–Study metric in ‘minimal block form’ (i.e. as close to diagonal as possible) which, specifically for the case of quadrilateralland, is given by use of Gibbons–Pope-type coordinates. Whilst they are not entirely diagonal, these are in a number of ways analogous to (ultra)spherical coordinates (this is an increase in complexity from triangle reminiscent of that in passing from the Schwarzschild metric to the Kerr metric, or in beginning to pass from diagonal Bianchi IX models to nondiagonal ones). Also, Key 9 (covered in more detail in [62], Paper II) is that the isometry group of  $\mathbb{CP}^2$  is  $SU(3)/\mathbb{Z}_3$  and  $SU(3)$  is well-known to have  $SU(2) \times U(1)$  subgroups; a particular choice of Gibbons–Pope-type coordinates then happens to be adapted to each such. Gibbons–Pope-type coordinates are a particularly hefty Key due to their subsequent usefulness in characterizing conserved quantities as expounded in Paper II and in separating the free-potential time-independent Schrödinger equation as expounded in Paper III. Serna and I lucidly interpreted Gibbons–Pope coordinates in quadrilateralland terms. As well as the H or K choice of tree type, the choice of inhomogeneous ratio coordinates affects the form of the quadrilateralland interpretation of these. E.g. (using Fig 1 and 8’s notation), the basic K = K(T)’s  $\beta$  is then the ratio of the back to the seat (i.e. a sharpness/flatness shape quantity for the obvious triangle subsystem), whilst  $\chi$  is the ratio of the back-and-seat (which in  $d > 1$  forms the obvious triangle subsystem) to the leg. Its  $\phi$  and  $\psi$  are a sum and a difference of knife angles.

On the other hand, K(M\*D)’s  $\beta$  is then the ratio of the seat to the leg, whilst its  $\chi$  is the ratio of the seat-and-leg to the back. Its  $\phi$  and  $\psi$  are a sum and a difference of a knife angle and a post angle.

for the third choice of ratios for the K case, and for the corresponding analysis of the H case, see Secs 7 and 9.3.

Key 4: Any  $N$ -a-gon's relationalspace is the cone over the corresponding shape space.

Key 5: Triangleland's useful Dragt-type shape quantities (ellipticity, anisoscelesness and four times the mass-weighted area per unit  $I$ ) to the case of the relational quadrilateral extend to quadrilateralland as follows. There is one ellipticity, one linear combination of ellipticities, three anisoscelesnesses and a quantity proportional to the square root of the sum of the squares of the three mass-weighted areas per unit  $I$ . In the case based on a Jacobi K choice of clustering, these quantities refer to three cluster-dependent 'coarse-graining triangles' [Fig 7.j)–l)], whilst in the Jacobi H case, these quantities refer to two of these and a 'coarse-graining' rhombus [Fig 7.e)–g)]; in each case, these are obtained from the quadrilateral by striking out each Jacobi vector in turn. The last of these quantities, is furthermore like the triangle case's  $4 \times \text{area}$  [=  $\text{demo}(N = 3)$ ] in being a democratic invariant (cluster-independent quantity),  $\text{demo}(N = 4)$ .

Key 6: triangleland's parabolic coordinates, that pick out the base and apex Jacobi magnitudes as separate subsystems alongside the relative angle between the corresponding Jacobi vectors, generalize in a number of ways to the quadrilateralland context as the Kuiper coordinates [79]. There are now 6 (a  $\mathbb{C}^3$ 's worth) of these, so they are now a partly-redundant formulation (i.e. with two relations upon them). They consist of magnitudes of the 3 mass-weighted Jacobi vectors and the 3 inner products between these vectors (i.e. relative angle variables). These are then physically interpreted in the quadrilateralland context as the three partial moments of inertia of the system and the three abovementioned anisoscelesnesses.

Key 7: for the simpler models considered hitherto, the shape space dimension was small enough to allow for complete graphical presentation of the interpretation of the regions of the shape space in RPM terms. E.g. 4-stop metroland and triangleland were both represented as distinct tessellations of their configuration space spheres [25] and Fig 4, and [53, 46], and Fig 5.  $\mathbb{CP}^2$  being 4-dimensional, one is more limited in visualization. Nevertheless, I provided a characterization of the RPM-significant submanifolds of this. I used Kuiper coordinates to determine the quadrilateralland counterpart of the split of triangleland's sphere into two hemispheres of mirror-image configurations that join along an equator of collinear configurations (which is one of [46]'s most important and useful results). One has now two topological  $\mathbb{S}^4$  'hemi- $\mathbb{CP}^2$ 's' that join along the ' $\mathbb{CP}^2$ -equator' of collinear configurations that form the mirror-image-identified 4-stop metroland and thus are a  $\mathbb{RP}^2$  submanifold. Upon codimension-1 embedding into a surrounding Euclidean space, this join maps into the Veronese surface, in accord with Kuiper's theorem. Quadrilateralland's  $\mathbb{CP}^2$  is additionally decorated by a net of 6  $\mathbb{S}^2$  trianglelands (in each case with one vertex being a double collision corresponding to the  $\binom{4}{2}$  pairs of particles coinciding). I also showed how  $\mathbb{S}^2$ 's corresponding to the coarse-graining 'trianglelands' (and 'rhombusland') occur within quadrilateralland, by applying restrictions on Gibbons–Pope-type coordinates.

I also used Gibbons–Pope-type and Kuiper coordinates to investigate the 6 possible labellings of square configurations and a weaker criterion of the cosmologically and quantum-cosmologically interesting topic of uniform states based on the extremization of  $\text{demo}(N = 4)$ . See Sec 12.3 for more on this, after I consider  $N$ -a-gonland extensions and Key 8: configuration space regions (which have a number of Problem of Time approach applications used in Papers III and IV).

## 13.2 $N$ -a-gonland generalization of this paper

Most of Keys 1 to 4 readily carry over to  $N$ -a-gonland.

The numbers and complexities of the qualitatively different types of Jacobi trees increase.

Key 2: the topological part is covered in Fig 10 for arbitrary  $N$ . The first part of Key 3's arbitrary- $N$  status should be contrasted with how the 3- $d$  models' shape spaces forming a much harder sequence [16], which gives a strong practical reason to study the 2- $d$  models (particularly since the analogy with geometrodynamics does not require dimension 3 in order to work well, as explicated in [7]).

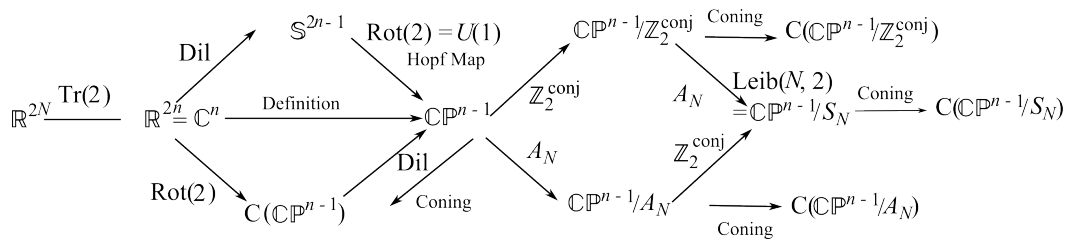


Figure 10: The general sequence of configuration spaces for  $N$ -a-gonland.

It is Key 3b that does not readily generalize. It remains to be seen whether the intrinsic Gibbons–Pope type coordinates can be extended to  $N$ -a-gonland in a way that maintains their usefulness as an answer to block-minimality, subgroup adaptation and subsequent usefulness in characterizing conserved quantities [58] and via separating the free-potential time-independent Schrödinger equation.

Of Keys 5 and 6, extending the Dragt/parabolic/shape/Kuiper type of redundant coordinates is itself straightforward, though it is not clear the extent to which the resulting coordinates will retain usefulness for the study of each  $\mathbb{CP}^{N-2}$ . Certainly the number of Kuiper-type coordinates (based on inner products of pairs of Jacobi vectors, of which there are

$N\{N - 1\}/2$  further grows away from  $2N = \dim(\mathbb{CP}^N)$  as  $N$  gets larger. There are now  $n - 1$  independent ellipticities, one configuration space radius  $\rho$ , one sum of squares of areas type of democracy invariant, but, additionally, the number of anisocleasnesses goes as  $n\{n - 1\}/2$ , so that the latter substantially overwhelms the shape space dimension. This suggests that these more general models will require a tighter set of shape variables than the present paper's one which includes all of its possible anisocleasness quantities.

As regards Key 7, the  $N$ -a-gonland significance of two half-spaces of different orientation separated by an orientationless manifold of collinearities gives reason for double covers to the  $N - 2 \geq 3$   $\mathbb{CP}^{N-2}$  spaces to exist for all  $N$ . However, there is no known guarantee [85] that these will involve geometrical entities as simple as or tractable as quadrilateralland's  $\mathbb{S}^4$  for the half-spaces, or of the Veronese surface  $V$  as the place of branching. It is clear that the manifold of collinearities within  $N$ -a-gonland's  $\mathbb{CP}^{N-2}$  itself is  $\mathbb{RP}^{N-2}$ , so at least that is a known and geometrically-simple result for the structure of the general  $N$ -a-gonland. Beyond the above, I leave  $N$ -a-gonland's significant submanifold structure as an open problem.

As regards the Quantum Information Theory counterpart of this work, one possible usefulness of representing qutrit states as quadrilaterals is via the convenience of having a  $2-d$  graphical representation, which, moreover, remains  $2-d$  as one passes to the study of qunits. In this picture, the relation between the 3 included  $SU(2)$  ladders and the 3 coarse-graining triangles (or two triangles and one parallelogram) is that there are 3 constituent (overlapping) qubits in a qutrit, and this pattern continues for higher  $n$ .

I finally comment that the large- $N$  limit of  $N$ -a-gonland should also be of interest, alongside the study of the statistical mechanics/entropy/notions of information/relative information/correlation that timeless approaches are concerned with.

### 13.3 Key 8: regions of configuration space and their uses in Problem of Time approaches

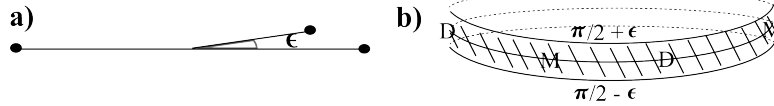


Figure 11: a)  $\epsilon$ -collinearity in space for three points. b) The corresponding configuration space belt of width  $2\epsilon$ .

Configuration space is useful classically and as a place where QM unfolds; the study so far in this paper needs, however, to be supplemented with a study of regions as well as of the submanifolds considered so far. Classically, this corresponds to approximate knowledge of configuration. One can pin a QM probability on that in some (part) timeless interpretations such as the Naïve Schrödinger Interpretation or Halliwell's approach.

I find that Gibbons–Pope type coordinates are useful in considering regions of configuration space: the volume element is simple in these,  $\sin^3 \chi \cos \chi d\chi \sin \beta d\beta d\phi d\psi/8$ , and so are characterizations for a number of physically and geometrically significant regions.

Application 1) To computing naïve Schrödinger interpretation probabilities of ‘being’, via (14). This is a continuation of what I have done in previous papers for metrolands and triangleland [25, 46, 22, 47]. Note the ‘sphere factor’ within this volume element (the  $\beta$  and  $\phi$  factors). The above gives

$$\text{Prob}(\text{Region } R) \propto \int_R |\Psi(\chi, \beta, \phi, \psi)|^2 \sin^3 \chi \cos \chi d\chi \sin \beta d\beta d\phi d\psi. \quad (82)$$

Then both volume element and the wavefunctions (at least in the free potential case [62, 59]) separate into a product of factors  $C(\chi)D(\beta)E(\phi)H(\psi)$ , and also the region of integration (at least for a number of physically-significant regions including the below examples) so that the integration over configuration space reduces to a product of  $1-d$  integrals, rendering integration relatively straightforward.

The next issue addressed in this paper is to characterize some physically-significant R's. [The wavefunctions themselves are provided in [59], where I combine them and this paper's study of regions to also provide naïve Schrödinger interpretation probabilities for approximate-collinearity, approximate-squareness and approximate-triangularity.]

4-stop metroland and triangleland counterparts of such regions are e.g. caps, belts and lunes in spherical polar coordinates [25, 46] endowed with particular physical significance, such as the cap of  $\epsilon$ -equilaterality or Fig 8b)'s belt of  $\epsilon$ -collinearity]. The present paper's regions are somewhat more involved due to involving the more complicated and higher-dimensional geometry of  $\mathbb{CP}^2$ . For the shape space spheres, I noted a correspondence between the size of the region in question (e.g. the radius of a small cap, the width of a small belt, the angle of a narrow lune and combinations of these by union, intersection and negation) and the size of the departure in space from the precise configuration. E.g. the width of the belt of collinearity on the shape sphere corresponds to a Kendall type [16, 46] notion of  $\epsilon$ -collinearity of three points in space (Fig 11a). Establishing such correspondences is then part of the study of physically-significant regions of the  $\mathbb{CP}^2$  configuration space also. In particular, I consider the following.

I) Approximately-collinear quadrilaterals. Exactly collinear was mathematically a  $\mathbb{RP}^2$  corresponding to, using the H-coordinates most natural here, both angular coordinates  $\phi$  and  $\psi$  being 0 or an integer multiple of  $\pi$ . This condition is now  $\epsilon$ -relaxed. Thus the region of integration is all values of the ratio coordinates  $\beta$  and  $\chi$  whilst the angular coordinates are to live within the following union of products of intervals:

$$(\{0 \leq \phi \leq \epsilon\} \cup \{\pi - \epsilon \leq \phi \leq \pi + \epsilon\} \cup \{2\pi - \epsilon \leq \phi \leq 2\pi\}) \times$$

$$(\{0 \leq \psi \leq \epsilon\} \cup \{\pi - \epsilon \leq \psi \leq \pi + \epsilon\} \cup \{2\pi - \epsilon \leq \psi \leq 2\pi + \epsilon\} \cup \{3\pi - \epsilon \leq \psi \leq 3\pi + \epsilon\} \cup \{4\pi - \epsilon \leq \psi \leq 4\pi\}) . \quad (83)$$

From the perspective of each configuration in space, this notion of  $\epsilon$ -collinearity corresponds to the relative angles  $\Phi_1$  and  $\Phi_2$  each lying within  $[0, \epsilon] \cup [\pi - \epsilon, \pi + \epsilon] \cup [2\pi - \epsilon, 2\pi]$ .

II) Quadrilaterals that are approximately triangular or approximately one of the mergers depicted in the coarse-graining triangles or rhombi of Fig 8. The exact configuration in each of these cases is a sphere characterized by one ratio variable and one relative angle variable. In moving away from exact triangularity, a second  $\epsilon$ -sized ratio variable becomes involved, and, in doing so one is rendering the other relative angle meaningful, allowing it to take all values. Thus the region of integration here is all angles, one ratio variable taking all possible values also, and the other being confined to an  $\epsilon$ -interval about the value that corresponds to exact triangularity. To be more concrete, consider the  $\{+43\}$  triangle in Gibbons–Pope type coordinates that derive from H-coordinates. Here,  $\beta = \pi$ , so the region of integration is all  $\phi$ , all  $\psi$ , all  $\chi$  and  $\pi - \epsilon \leq \beta \leq \pi$ . Then, for example, the notion of  $\epsilon$ - $\{+43\}$  triangular corresponds in space to the ratio (post 1)/crossbar. being of size  $\epsilon/2$  or less.

III) One of the labellings of exact square configuration is at  $\phi = 0$ ,  $\psi = \pi$ ,  $\beta = \pi/2$  and  $\chi = \pi/4$ . Approximate squareness allows for all four of these quantities to be  $\epsilon$ -close, so that the region of integration is a product of four intervals of width  $\epsilon$  about these points. That corresponds to the two sides of the Jacobi H being allowed to be  $\epsilon$ -close ( $\beta$ -variation), the quadrilateral to vary in height to length ratio ( $\chi$ -variation) and for each of the sides of the Jacobi H to become non-right with respect to the crossbar ( $\psi$ - and  $\phi$ -variation), which is an entirely intuitive parametrization of the possible small departures from exact squareness. [If one is interested in all of the labellings of the square, one can straightforwardly characterize each with a similar construction and again take the union of these regions.] This example is clearly furthermore useful as a notion of approximate uniformity, which is of interest in Classical and Quantum Cosmology.

Application 2) The above notions of closeness and use of the Fubini–Study kinetic metric additionally embody control over localization both in space and in configuration space, allowing for the triangleland work towards establishing a records theory in [7] (concerning in particular notions of distance between shapes) to be extended to quadrilateralland. Records are usually envisaged as localized subsystems, by which having a certain richness of these available in one’s model is conducive to a nontrivial study of this locality, associated notions of information and the emergence of a semblance of dynamics involving nontrivial structure formation. As argued in the Introduction and in Sec 13.1, quadrilateralland is rather richer as regards constituent subsystems than triangleland is, whilst retaining a nontrivial linear constraint (that parallels the GR momentum constraint) in the quantum-cosmologically relevant scaled case.

Application 3) In the semiclassical approach, the many approximations used only hold in certain regions, in particular the crucial Born–Oppenheimer and WKB approximations. The semiclassical application awaits having the semiclassical approach sorted out for scaled quadrilateralland [59]; for the moment see [35] for the triangleland counterpart of such workings. This is needed for extending semiclassical work in [22, 47, 35] to the quadrilateralland setting.

Application 4) Regions also feature in the histories approach and in Halliwell’s combination of this with semiclassical approach and timeless ideas. Indeed, an extra connection between these approaches is that [42, 43] the semiclassical approach aids in the computation of timeless probabilities of histories entering given configuration space regions. This, by the WKB assumption, gives a semiclassical flux into each region in terms of  $S(h)$  and, in a simple case, the Wigner function (see e.g. [42]). Moreover, such schemes go beyond the standard semiclassical approach, so there is some chance that further objections to the semiclassical approach (problems inherited from the Wheeler–DeWitt equation and problems with reconstructing spacetime in such approaches) would be absent from the new unified strategy. I am presently considering [26] extending [42] to RPM’s (for the moment for triangleland), looking at, without reference to time, what is the probability of finding the system in a series of regions of configuration space for a given eigenstate of the Hamiltonian [42]? Halliwell studied this with a free particle, a working which has a direct, and yet more genuinely closed-universe, counterpart for scaled triangleland [6] via the ‘Cartesian to Dragt coordinates analogy’ that can be seen between Secs 5.2 and 5.3 allowing me to transcribe this working to a relational context. The quadrilateralland extension of this calculation is then desirable for its combination of further physical modelling features and mathematical novelty as a histories theory.

I end by noting that these applications are revisited in substantially more detail in Papers III and IV.

**Acknowledgements:** I thank: my wife Claire, Amelia, Sophie, Sophie, Anya, Hettie, Hannah, Bryony, Amy and Emma for being supportive of me whilst this work was done. Mr Eduardo Serna for discussions and reading the manuscript and some of thinking behind Sec 7. Professors Don Page and Gary Gibbons for teaching me about  $\mathbb{CP}^2$  in 2005 and 2007. Dr Julian Barbour for introducing me to RPMs in 2001. Professors Jonathan Halliwell, Chris Isham and Karel Kuchař for discussions. Professors Belen Gavela, Marc Lachièze-Rey, Malcolm MacCallum, Don Page, Reza Tavakol, and Dr Jeremy Butterfield for support with my career. Fqxi grant RFP2-08-05 for travel money whilst part of this work was done in 2009-2010, and Universidad Autonoma de Madrid for funding in 2010-2011. The final drafting of this work was funded by a grant from the Foundational Questions Institute (FQXi) Fund, a donor-advised fund of the Silicon Valley Community Foundation on the basis of proposal FQXi-RFP3-1101 to the FQXi. I thank also Theiss Research and the CNRS for administering this grant.

# References

- [1] J.B. Barbour and B. Bertotti, Proc. Roy. Soc. Lond. **A382** 295 (1982).
- [2] J.B. Barbour, Class. Quantum Grav. **20** 1543 (2003), gr-qc/0211021.
- [3] J.B. Barbour, Class. Quantum Grav. **11** 2853 (1994).
- [4] J.B. Barbour, *The End of Time* (Oxford University Press, Oxford 1999).
- [5] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge 2004).
- [6] E. Anderson, Class. Quantum Grav. **26** 135020 (2009), arXiv:0809.1168.
- [7] E. Anderson, arXiv:1111.1472
- [8] E. Anderson, “Relationalism”, forthcoming; “From Mach’s ‘Time is Abstracted from Change’ to Timeless Approaches to the Problem of Time”, forthcoming; “Supersymmetry Versus Relationalism” , forthcoming.
- [9] C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto 1949).
- [10] E. Anderson, Class. Quantum Grav. **25** 175011 (2008), arXiv:0711.0288.
- [11] J.B. Barbour, B.Z. Foster and N. Ó Murchadha, Class. Quantum Grav. **19** 3217 (2002), gr-qc/0012089; E. Anderson, Gen. Rel. Grav. **36** 255, gr-qc/0205118; Phys. Rev. **D68** 104001 (2003), gr-qc/0302035; “Geometrodynamics: Spacetime or Space?” (Ph.D. Thesis, University of London 2004), gr-qc/0409123; in *General Relativity Research Trends, Horizons in World Physics* **249** ed. A. Reimer (Nova, New York 2005), gr-qc/0405022; Stud. Hist. Phil. Mod. Phys. **38** 15 (2007), gr-qc/0511070; in “Classical and Quantum Gravity Research”, ed. M.N. Christiansen and T.K. Rasmussen (Nova, New York 2008), arXiv:0711.0285.
- [12] E. Anderson, Class. Quantum Grav. **25** 025003 (2008), arXiv:0706.3934.
- [13] E. Anderson, arXiv:1001.1112.
- [14] R.F. Baierlain, D. Sharp and J.A. Wheeler, Phys. Rev. **126** 1864 (1962).
- [15] E. Anderson, Class. Quantum Grav. **24** 2935 (2007), gr-qc/0611007.
- [16] D.G. Kendall, D. Barden, T.K. Carne and H. Le, *Shape and Shape Theory* (Wiley, Chichester 1999).
- [17] K.V. Kuchař, in *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics* ed. G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore 1992).
- [18] J.B. Barbour, in *Decoherence and Entropy in Complex Systems (Proceedings of the Conference DICE, Piombino 2002)* ed. H-T. Elze, Springer Lecture Notes in Physics 2003), gr-qc/0309089; E. Anderson, *Proceedings of 2003 Marcel Grossman Meeting, Rio de Janeiro*, arXiv:gr-qc/0312037; AIP Conf. Proc. **861** 285 (2006), gr-qc/0509054; Class. Quantum Grav. **27** 045002 (2010), arXiv:0905.3357; S.B. Gryb, Phys. Rev. **D81** 044035 (2010), arXiv:0804.2900; Class. Quantum Grav. **26** (2009) 085015, arXiv:0810.4152; J.B. Barbour and B.Z. Foster, arXiv:0808.1223; J.B. Barbour, arXiv:1105.0183.
- [19] See e.g. C. Kiefer, *Quantum Gravity* (Clarendon, Oxford 2004).
- [20] E. Anderson, Class. Quantum Grav. **23** 2469 (2006), gr-qc/0511068.
- [21] E. Anderson, Class. Quantum Grav. **23** 2491 (2006), gr-qc/0511069.
- [22] E. Anderson, Class. Quantum Grav. **28** 065011 (2011), arXiv:1003.1973.
- [23] E. Anderson, for Proceedings of Paris 2009 Marcel Grossman Meeting, in Press, arXiv:0908.1983.
- [24] E. Anderson, Int. J. Mod. Phys. **D18** 635 (2009), arXiv:0709.1892; in *Proceedings of the Second Conference on Time and Matter*, ed. M. O’Loughlin, S. Stanič and D. Veberič (University of Nova Gorica Press, Nova Gorica, Slovenia 2008), arXiv:0711.3174.
- [25] E. Anderson and A. Franzen, Class. Quantum Grav. **27** 045009 (2010), arXiv:0909.2436.
- [26] E. Anderson, “Approaching the Problem of Time with a Combined Semiclassical-Records-Histories Scheme”, forthcoming.
- [27] M. Ryan, *Hamiltonian Cosmology* (Lecture Notes in Physics 13) (Springer, Berlin, 1972); J.B. Hartle and S.W. Hawking, Phys. Rev. **D28** 2960 (1983); D.L. Wiltshire, in *Cosmology: the Physics of the Universe* ed. B. Robson, N. Visvanathan and W.S. Woolcock (World Scientific, Singapore 1996), gr-qc/0101003.
- [28] S. Carlip, *Quantum Gravity in 2 + 1 Dimensions* (Cambridge University Press, Cambridge 1998).
- [29] C.J. Isham, in *Integrable Systems, Quantum Groups and Quantum Field Theories* ed. L.A. Ibort and M.A. Rodríguez (Kluwer, Dordrecht 1993), gr-qc/9210011.
- [30] J.J. Halliwell and S.W. Hawking, Phys. Rev. **D31**, 1777 (1985).
- [31] K.V. Kuchař, in *Quantum Gravity 2: a Second Oxford Symposium* ed. C.J. Isham, R. Penrose and D.W. Sciama (Clarendon, Oxford 1981); in *The Arguments of Time*, ed. J. Butterfield (Oxford University Press, Oxford 1999).
- [32] E. Anderson, in *Classical and Quantum Gravity: Theory, Analysis and Applications* ed. V.R. Frignanni (Nova, New York 2011), arXiv:1009.2157.
- [33] P.A.M. Dirac, Rev. Mod. Phys. **21** 392 (1949); C. Rovelli and L. Smolin, Nu. Phys. **B331** 80 (1990).



- [34] B.S. DeWitt, Phys. Rev. **160** 1113 (1967); V.G. Lapchinski and V.A. Rubakov, Acta Physica Polonica **B10** (1979); T. Banks, Nu. Phys. **B249** 322 (1985).
- [35] E. Anderson, Class. Quant. Grav. **28** 185008 (2011), arXiv:1101.4916.
- [36] E. Anderson, “Variational Underpinning for the Semiclassical Approach to Quantum Cosmology”, forthcoming.
- [37] S.W. Hawking and D.N. Page, Nu. Phys. **B264** 185 (1986); W. Unruh and R.M. Wald, Phys. Rev. **D40** 2598 (1989).
- [38] D.N. Page and W.K. Wootters, Phys. Rev. **D27**, 2885 (1983).
- [39] M. Gell-Mann and J.B. Hartle, Phys. Rev. **D47** 3345 (1993).
- [40] J.J. Halliwell, Phys. Rev. **D60** 105031 (1999), quant-ph/9902008.
- [41] J.B. Hartle, in *Gravitation and Quantizations: Proceedings of the 1992 Les Houches Summer School* ed. B. Julia and J. Zinn-Justin (North Holland, Amsterdam 1995), gr-qc/9304006.
- [42] J.J. Halliwell, in *The Future of Theoretical Physics and Cosmology (Stephen Hawking 60th Birthday Festschrift Volume)* ed. G.W. Gibbons, E.P.S. Shellard and S.J. Rankin (Cambridge University Press, Cambridge 2003), gr-qc/0208018.
- [43] J.J. Halliwell, Phys. Rev. **D80** 124032 (2009), arXiv:0909.2597; J.J. Halliwell, J. Phys. Conf. Ser. **306** 012023 (2011), arXiv:1108.5991.
- [44] See e.g. C. Marchal, *Celestial Mechanics* (Elsevier, Tokyo 1990).
- [45] E. Anderson, Class. Quantum Grav. **26** 135021 (2009), gr-qc/0809.3523.
- [46] E. Anderson, Gen. Rel. Grav. **43** 1529 (2011), arXiv:0909.2439.
- [47] E. Anderson, arXiv:1005.2507.
- [48] E. Anderson, Class. Quantum Grav. **24** 5317 (2007), gr-qc/0702083.
- [49] W.-Y. Hsiang and E. Straume, arXiv:math-ph/0609084, math-ph/0609076, math-ph/0608060.
- [50] R. Montgomery, Nonlin. **9** 1341 (1996), math/9510005; **11** 363 (1998).
- [51] R. Montgomery, Ergod. Th. Dynam. Sys. **25** 921 (2005), math/0405014.
- [52] R.G. Littlejohn and M. Reinsch, Phys. Rev. **A52** 2035 (1995).
- [53] D.G. Kendall, Statistical Science **4** 87 (1989).
- [54] A.J. Dragt, J. Math. Phys. **6** 533 (1965).
- [55] R. Montgomery, Arch. Rat. Mech. Anal. **164** 311 (2002).
- [56] W. Zickendraht, Phys. Rev. **159** 1448 (1967); J. Math. Phys. **10** 30 (1969); **12** 1663 (1970); R.G. Littlejohn and M. Reinsch, Rev. Mod. Phys. **69** 213 (1997).
- [57] V. Aquilanti, S. Cavalli and G. Grossi, J. Chem. Phys. **85** 1362 (1986).
- [58] E. Anderson, “Relational Quadrilateralland. II. Analogues of Isospin and Hypercharge”, forthcoming.
- [59] E. Anderson et al., “Relational Quadrilateralland. III. The Quantum Theory”, forthcoming.
- [60] E. Anderson, “Relational Quadrilateralland. IV. Problem of Time Applications”, forthcoming.
- [61] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Books, Reading, Massachusetts 1995).
- [62] A.J. MacFarlane, J. Phys. A: Math. Gen. **36** 7049 (2003).
- [63] A.J. MacFarlane, J. Phys. A: Math. Gen. **36** 9689 (2003); Nu. Phys. **B152** 145 (1979).
- [64] M. Berger, P. Gauduchon and E. Ozet, *Le Spectre d’une Variété Riemannienne. Lecture Notes in Mathematics* **194** (Springer, Berlin 1971).
- [65] N.P. Warner, Proc. Roy. Soc. Lond. **1383** 217 (1982).
- [66] See e.g. C.M. Caves and G.J. Milburn, quant-ph/9910001; A.B. Klimov, L.L. Sánchez-Soto, H. de Guise and G. Bjork, J. Phys. A **37** 4097 (2004), quant-ph/0312127; A.T. Böllükbası and T. Dereli, quant-ph/0511111.
- [67] See e.g. W. Zakrewski, *Low-Dimensional Sigma Models* (Hilger, Bristol 1989).
- [68] V. Golo and A. Perelomov, Phys. Lett. **79B** 112 (1978); H. Eischenherr, Nu. Phys. **B** 146 (1978).
- [69] G.W. Gibbons and C.N. Pope, Commun. Math. Phys. **61** 239 (1978); C.N. Pope, Phys. Lett. **97B** 417 (1980).
- [70] G.W. Gibbons and S.W. Hawking, Commun. Math. Phys. **66** 291 (1979); D.N. Page, arXiv:0912.4922
- [71] See e.g. S. Huggett, lecture notes from 2005 Twistor String Theory Conference.
- [72] E. Anderson, arXiv:1009.2161.
- [73] E. Anderson, arXiv:1102.2862.

- [74] The following consider a weighted projective space  $\mathbb{WCP}^3 = \mathbb{CP}^3/\mathbb{Z}_2$  in the context of M-theory. E. Witten, hep-th/0108165; M. Atiyah and E. Witten, hep-th/0107177; B.S. Acharya and E. Witten, hep-th/0109152; B.S. Acharya, in *Strings and Geometry Proceedings of the Clay Mathematics Institute 2002 Summer School* ed. M. Douglas, J. Gauntlett and M. Gross (American Mathematical Society, Providence, Rhode Island 2003), available online at <http://www.claymath.org/library/proceedings/cmip03c.pdf>; D. Joyce, *ibid*, mathDG/9910002; A. Collinucci, JHEP 0908:076 (2009), arXiv:0812.0175. mentions a  $\mathbb{WCP}^4$ ; R. Auzzi, M. Shifman and A. Yung Phys. Rev. **D73** 105012 (2006); Erratum-*ibid*. **D76** 109901 (2007), hep-th/0511150 also consider  $\mathbb{WCP}^5$ ; E. Witten, Adv. Theor. Math. Phys. **5** 841 (2002) hep-th/0006010 considers a  $\mathbb{WCP}^k$  and M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, Phys. Rev. **D74** 065021 (2006), hep-th/0607070 state that some such spaces are singular.
- [75] J.A. Wheeler, in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics* ed. C. DeWitt and J.A. Wheeler (Benjamin, New York 1968).
- [76] J.W. York, Phys. Rev. Lett. **28** 1082 (1972); J. Math. Phys. **14** 456 (1973); Ann. Inst. Henri Poincaré **21** 319 (1974); E. Anderson, J.B. Barbour, B.Z. Foster and N. Ó Murchadha, Class. Quantum Grav. **20** 157 (2003), gr-qc/0211022. E. Anderson, J.B. Barbour, B.Z. Foster, B. Kelleher and N. Ó Murchadha, Class. Quantum Grav **22** 1795 (2005), gr-qc/0407104.
- [77] J. Barbour and N. Ó Murchadha, arXiv:1009.3559.
- [78] A.E. Fischer and V. Moncrief, Gen. Rel. Grav. **28**, 221 (1996).
- [79] N.H. Kuiper, Math. Ann. **208** 175 (1974).
- [80] A. Trautman Int. J. Theor. Phys. **16** 561 (1977).
- [81] See e.g. J. Harris, *Algebraic Geometry, A First Course*, (Springer-Verlag, New York 1992).
- [82] C.J. Isham, in *Relativity, Groups and Topology II* ed. B.S. DeWitt and R. Stora (North-Holland, Amsterdam 1984).
- [83] N.P. Warner, Proc. Roy. Soc. Lond. **1383** 207 (1982).
- [84] See e.g. A.F. Beardon, *Primer on Riemann Surfaces* (Cambridge University Press, Cambridge 1984).
- [85] S. Kuroki, J. Math. Sci. **146** 5518 (2007).